Lecture 18 (2/25/2019)

Recall: the geometric series $1+q+q^2+q^3+q^4+\cdots$ can be written using Σ notation as

$$\sum_{k=0}^{\infty} q^k \quad \left(\text{or } \sum_{n=0}^{\infty} q^n \text{, the name of index can be chosen arbitrarily}\right)$$

The n'th partial sum $S_n = \underbrace{1+q+q^2+\cdots+q^{n-1}}_{\text{sum of the first } n \text{ terms of the series}}$

$$S_n = \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

If $-1<q<1$, $q^n \to 0$. Thus $S_n \to \frac{1}{1-q}$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \text{(convergent series)}$$

If $q=-1$,

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

S_n doesn't converge

In general, if $q \le -1$ or $q \ge 1$, the series diverges

* Observation: if one modifies finitely many terms of a series, the convergence / divergence doesn't change.
  This observation is often used in practice: there are times when the first few terms of the series doesn't fall into pattern. One can ignore those terms when considering convergence/divergence.

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k$$

Algebra of series:

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then

- $\sum_{n=1}^{\infty} c a_n$ converges and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

constant not depending on n

- $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges and $\sum_{n=1}^{\infty} (a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Why?

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Then take $n \to \infty$.

$$(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)+\cdots$$
$$= (a_1+a_2+a_3+a_4+\cdots) + (b_1+b_2+b_3+b_4+\cdots)$$

Rearrangement of terms is possible if each infinite sum converges.

In general, an infinite sum is <u>not commutative</u>.

<u>Ex</u>:

$a_k = 1$ for all k

$b_k = -1$ for all k

a_k: $1, 1, 1, 1, 1, \ldots$ $\qquad \sum_{k=1}^{\infty} a_k = \infty$

b_k: $-1, -1, -1, -1, \ldots$ $\qquad \sum_{k=1}^{\infty} b_k = -\infty$

$a_k + b_k$: $0, 0, 0, 0, \ldots$ $\qquad \sum_{k=1}^{\infty} (a_k + b_k) = 0$

Thm: If $a_k \geq 0$ for all k then the infinite sum $\sum_{k=1}^{\infty} a_k$ doesn't change when one reorder the terms.

Ex:

$$\ln x = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots$$

Taylor series of the logarithm function about base point 1.

$$\ln 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\neq \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right)$$

in fact, both of these sum diverge (we'll discuss next time why)

* Observation:

- If $a_n \geq 0$ and the partial sum S_n is bounded from above then the series $\sum_{n=1}^{\infty} a_n$ converges.

Why? S_n is an increasing sequence that is bounded from above. Thus, S_n converges.

Ex:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\frac{1}{k^2} < \frac{1}{k^2 - k} = \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k} \quad \text{for any } k \geq 2$$

Apply for $k = 2, 3, 4, \ldots$

$$\frac{1}{2^2} < \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}$$

$+$

$$\frac{1}{4^2} < \frac{1}{3} - \frac{1}{4}$$

$$\cdots\cdots$$

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$$

$$\frac{1}{2^2} + \cdots + \frac{1}{n^2} < 1 - \frac{1}{n} < 1$$

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n} < 2$$

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges