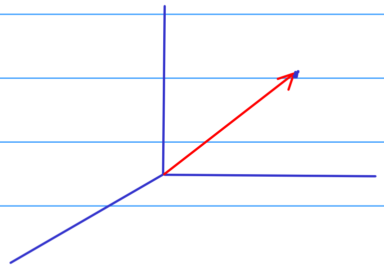
Lecture 2 (1/9/2019)

$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) :$ each x_i is a real number$\}$

$\mathbb{R}^n$ is call a vector space, or more specifically n-dimensional vector space.

Each element is called a vector. A vector can be viewed as a point. A point can be viewed as a vector based at the origin.



One can add two vectors (component by component), and scale a vector by a number (scale each component)

$$\left.\begin{aligned} x &= (x_1, x_2, \dots, x_n) \\ y &= (y_1, \dots, y_n) \\ c &\in \mathbb{R} \end{aligned}\right\} \Rightarrow \begin{aligned} x+y &= (x_1+y_1, \dots, x_n+y_n) \\ cx &= (cx_1, cx_2, \dots, cx_n) \end{aligned}$$

A number is a vector (when $n=1$).

One can talk about maps between vector spaces.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

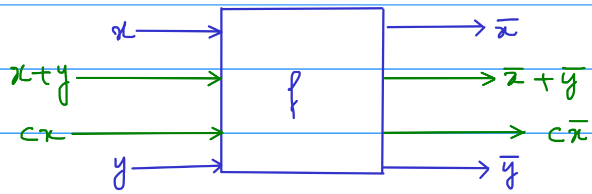
A type of maps/functions/transformations (these terms used interchangeably) that is compatible with vector spaces is linear maps.

f is said to be a linear map if:

- Additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.
- Scalar multiplicative: $f(cx) = cf(x)$ for all $c \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Note: if $x = (x_1, x_2, \dots, x_n)$, we will write

$$f(x) = f(x_1, x_2, \dots, x_n)$$
$$0 = (0, 0, \dots, 0)$$



Ex:

Any linear map $f: \mathbb{R} \to \mathbb{R}$ must have the form $f(x) = ax$, where a is a constant.

Why? $f(x) = f(\underset{\substack{\text{scalar}\\ x}}{x}\cdot\underset{\text{vector } 1}{1}) = x\,\underbrace{f(1)}_{a} = ax$

Ex:

$f(x) = x^2$ is <u>not</u> a linear map because it is not additive

$$(1+2)^2 \neq 1^2 + 2^2$$

Ex:

$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1 + 2x_2$$

One can check that f is both additive and scalar mult.

$$f(x_1+y_1, x_2+y_2) = f(x_1, x_2) + f(y_1, y_2)$$
$$f(cx_1, cx_2) = cf(x_1, x_2)$$

Thus, f is linear.

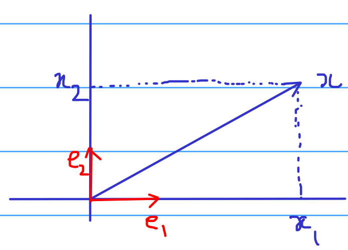
Ex: $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = \sqrt[3]{x_1}\,\sqrt[3]{x_2^2}$

This map is scalar multiplicative, but not additive.

$$f(1,1) \neq f(1,0) + f(0,1)$$

Ex: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is linear.

$$x = (x_1, x_2) = x_1 e_1 + x_2 e_2$$



$$f(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$$
$$= x_1 \underbrace{f(e_1)}_{a} + x_2 \underbrace{f(e_2)}_{b}$$

$f(x_1, x_2) = ax_1 + bx_2$, a, b are constants.

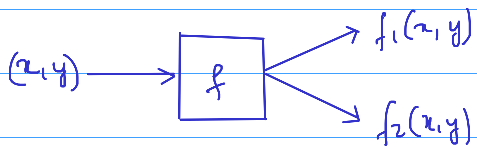
Any linear map from $\mathbb{R}^2$ to $\mathbb{R}$ must be of this form.

Ex: Similarly, any linear map from $\mathbb{R}^n$ to $\mathbb{R}$ must be of the form $f(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ where $a_1, a_2, \dots, a_n$ are constants.

Ex: Consider a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

f takes 2 inputs and yields 2 outputs: $f = (f_1, f_2)$



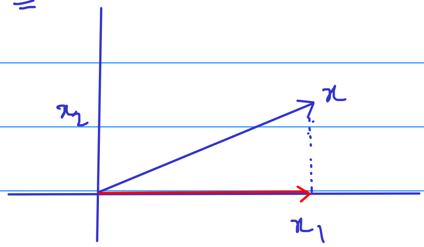
One can check that f_1 and f_2 are linear maps from $\mathbb{R}^2$ to $\mathbb{R}$.

$$f_1(x_1, x_2) = a x_1 + b x_2$$

$$f_2(x_1, x_2) = c x_1 + d x_2$$

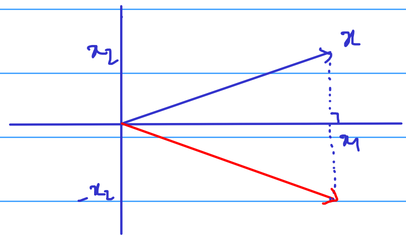
$$\Rightarrow f(x_1, x_2) = (a x_1 + b x_2, c x_1 + d x_2).$$

Ex: The projection map $f(x_1, x_2) = (x_1, 0)$.



$$\begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 0 \end{cases}$$

Ex: The reflection map $f(x_1, x_2) = (x_1, -x_2)$



$$\begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = -1 \end{cases}$$