Lecture 20 (3/1/2019)

More examples on Comparison test:

Ex: $\sum \sin\left(\frac{1}{\sqrt{n}}\right)$

When x is close to zero, $\frac{\sin x}{x} \approx 1$ because $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

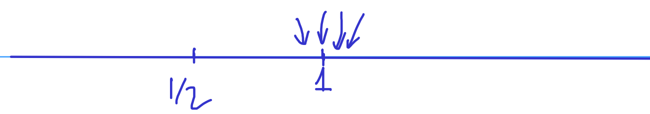
$$\sin\left(\frac{1}{\sqrt{n}}\right) \sim \frac{1}{\sqrt{n}}$$

Then $\sum \sin\left(\frac{1}{\sqrt{n}}\right) \sim \sum \frac{1}{\sqrt{n}}$ which diverges

Guess: the series diverges.

How to prove it? use Comparison test

$\frac{\sin x}{x} > \frac{1}{2}$ if x is sufficiently close to 0



$\Rightarrow \frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{1/\sqrt{n}} > \frac{1}{2}$ for n large ------- $\sin\left(\frac{1}{\sqrt{n}}\right) > \frac{1}{2}\frac{1}{\sqrt{n}}$

The series $\sum \frac{1}{2}\frac{1}{\sqrt{n}}$ diverges. So does $\sum \sin\left(\frac{1}{\sqrt{n}}\right)$

Ex: $\sum \sin\left(\frac{1}{n^2}\right)$ converges (similar method, compare with $\frac{2}{n^2}$)

$$\frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} \approx 1 \quad \text{for } n \text{ large}$$

Thus, $\frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} < 2$ for n large

$$\sin\left(\frac{1}{n^2}\right) < \frac{2}{n^2} \quad \text{for } n \text{ large}$$

The series $\sum \frac{2}{n^2}$ converges, so does $\sum \sin\left(\frac{1}{n^2}\right)$

Ex: $\sum n \sin\left(\frac{1}{n}\right)$

For n large, $\sin\left(\frac{1}{n}\right)$ small.

It's not clear whether $n \sin\left(\frac{1}{n}\right)$ is large or small.

Analyzing: Using calculator, we see that this quantity is about 1 when n is large, indicating that the series doesn't converge.

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'Hospital}}{=\!=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Thus,

$$\lim_{n\to\infty} \underbrace{\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}}_{n \sin\frac{1}{n}} = 1$$

Because the general term of the series doesn't converge to 0, the series doesn't converge.

Integral test

Recall that we used tricks to show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad \text{diverges}$$

These series are examples of a general class of series of type

$$a_k = f(k)$$

where f is some function. For example, the first series corresponds to $f(x) = \frac{1}{x^2}$, the second corresponding to $f(x) = \frac{1}{\sqrt{x}}$.

Integral test

If f is a $\begin{cases} \text{continuous} \\ \text{nonnegative} \\ \text{decreasing} \end{cases}$ for x large ($x \geq N$ for some N)

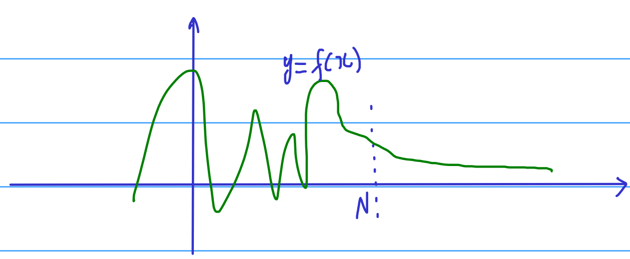
Then

$$\sum_{k=1}^{\infty} a_k \text{ converges if } \int_N^{\infty} f(x)\,dx < \infty$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges if } \int_N^{\infty} f(x)\,dx = \infty$$

Here $\int_N^{\infty} f(x)\,dx$ is called an improper integral, and is defined as

$$\lim_{M \to \infty} \int_N^M f(x)\,dx$$



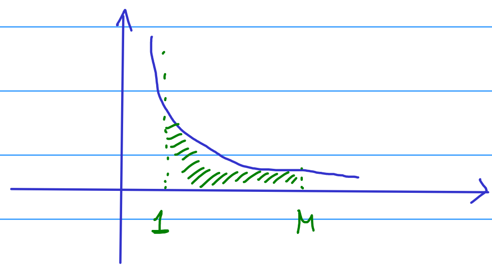
One is only interested in the long-term behavior of f.

Ex: $\sum \frac{1}{n^p}$ $(p > 0)$ ... This is called p-series

For $p = 1$, it is called harmonic series.

$a_n = \frac{1}{n^p} = n^{-p} = f(n)$

where $f(x) = x^{-p}$.



This function is $\begin{cases} \text{continuous when } x \geq 1 \\ \text{nonnegative} \\ \text{decreasing} \end{cases}$

$$\int_1^M f(x)\,dx = \int_1^M x^{-p}\,dx \overset{p \neq 1}{=\!=} \left.\frac{x^{1-p}}{1-p}\right|_1^M$$

$$= \frac{M^{1-p} - 1}{1-p}$$

- If $p > 1$ then $M^{1-p} \to 0$ as $M \to \infty$. Thus,

$$\int_1^\infty x^{-p}\,dx = \frac{-1}{1-p} = \frac{1}{p-1} < \infty$$

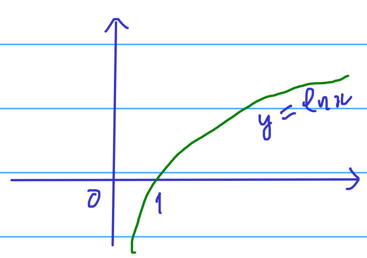
The series converges.

- If $p < 1$ then $M^{1-p} \to \infty$ as $M \to \infty$. Thus,

$$\int_1^\infty x^{-p}\,dx = \infty$$

The series diverges.

- If $p = 1$ then $\int_1^M x^{-1}\,dx = \ln M \longrightarrow \infty$ as $M \to \infty$



Conclusion: