Lecture 21 (3/4/2019)

* Observation:

If $\sum |a_n|$ converges (i.e. $< \infty$) then $\sum a_n$ also converges.

why? Comparison test

$$\begin{cases} |a_n| \le b_n \\ \sum b_n < \infty \end{cases} \quad \text{with } b_n = |a_n|$$

The converse is not true:

The series $\sum \frac{(-1)^n}{n}$ is converges because

$$\sum \frac{(-1)^n}{n} = \frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

$$= -\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right)$$

$$= -\ln 2$$

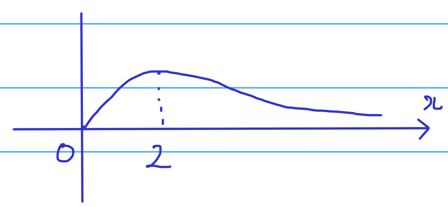
However, $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n} = \infty$ (harmonic series)

Ex: $\sum (-1)^n n^2 e^{-n}$

Let's try to verify that this series converges.

Suffices to show that $\sum n^2 e^{-n}$ converges.

$f(x) = x^2 e^{-x}$



f is cont. and nonnegative.

We need to check if f is decre.

$$f'(x) = 2x e^{-x} - x^2 e^{-x}$$

$$= (2x - x^2) e^{-x}$$

$$= x(2-x) e^{-x}$$

$$< 0 \quad \text{if } x > 2$$

f is decreasing on the interval $(2, \infty)$.

We only need to check if $\int_2^\infty f(x)\,dx < \infty$

$$\int_2^M x^2 e^{-x}\,dx \qquad \begin{cases} u = x^2 \\ dv = e^{-x}\,dx \end{cases} \quad \cdots\cdots \quad \begin{cases} du = 2x\,dx \\ v = -e^{-x} \end{cases}$$

int. by parts

$$-x^2 e^{-x}\Big|_2^M + \int_2^M 2x\,e^{-x}\,dx$$

$$= -M^2 e^{-M} + 2^2 e^{-2} + \int_2^M 2x\,e^{-x}\,dx$$

$$\begin{cases} u = 2x \\ dv = e^{-x}\,dx \end{cases} \quad \cdots\cdots \quad \begin{cases} du = 2dx \\ v = -e^{-x} \end{cases}$$

int. by parts

$$= -M^2 e^{-M} + 2^2 e^{-2} - 2x\,e^{-x}\Big|_2^M + \int_2^M 2e^{-x}\,dx$$

$$= -M^2 e^{-M} + 2^2 e^{-2} - 2Me^{-M} + 2(2)\,e^{-2} - 2e^{-x}\Big|_2^M$$

$$= \underbrace{-M^2 e^{-M}}_{\to 0} + 2^2 e^{-2} \underbrace{-2Me^{-M}}_{\to 0} + 2(2)\,e^{-2} \underbrace{-2e^{-M}}_{\to 0} + 2e^{-2}$$

As $M \to \infty$:

The limit as $M \to \infty$ exists. Thus, $\int_2^\infty f(x)\,dx < \infty$.

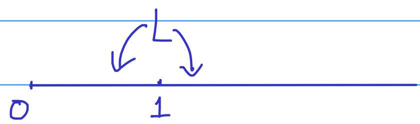
The series converges.

Ratio test

Suppose $a_n \neq 0$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

- If $L > 1$, $\sum a_n$ diverges
- If $L < 1$, $\sum a_n$ converges (absolutely)
- If $L = 1$, test fails.

why is it true?

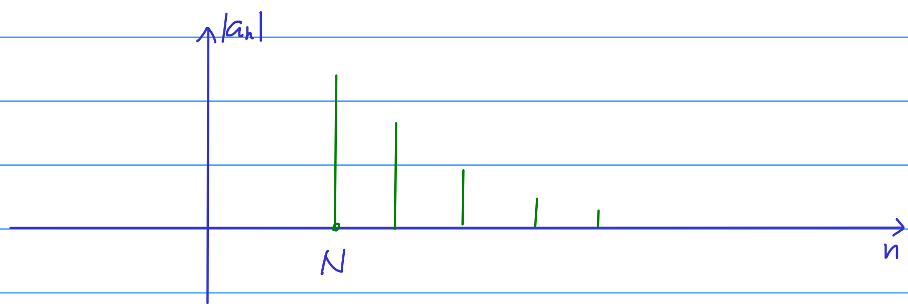


$$\frac{|a_{n+1}|}{|a_n|} \approx L \text{ for } n \text{ large}$$

$$|a_{n+1}| \approx L|a_n| \quad \text{for } n \text{ large}$$

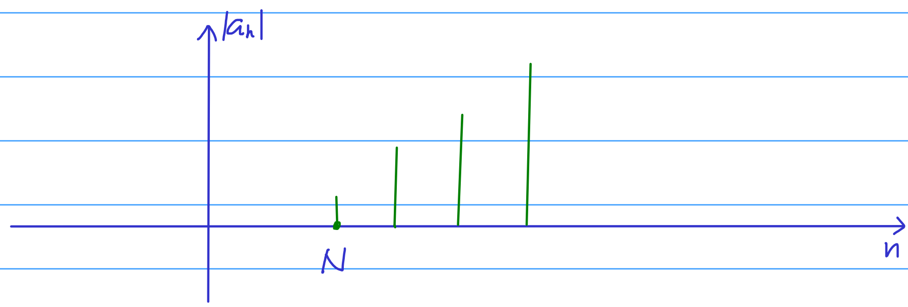
$$|a_{n+1}| \approx L|a_n| \approx L\,L|a_{n-1}| \approx \cdots \approx L^n|a_1|$$

If $L<1$, $\sum a_n$ is dominated by geometric series $\sum L^n a_1$, which converges.



If $L>1$, $|a_{n+1}| \approx L|a_n| > |a_n|$

a_n doesn't goes to 0 as $n\to\infty$



Ex :

$$\sum \frac{n^2}{2^n}$$

$$\left.\begin{aligned} a_n &= \frac{n^2}{2^n} > 0 \\ a_{n+1} &= \frac{(n+1)^2}{2^{n+1}} \end{aligned}\right\} \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \frac{2^n}{2^{n+1}} = \left(\frac{n+1}{n}\right)^2 \frac{1}{2}$$

$$= \left(1 + \frac{1}{n}\right)^2 \frac{1}{2}$$

$$\xrightarrow{n \to \infty} \frac{1}{2} < 1$$

Conclusion : $\sum \frac{n^2}{2^n}$ converges.