Lecture 22 (3/6/2019)

More examples on Ratio test:

Ex: $\sum \frac{n^2}{n!}$

General term $\quad a_n = \frac{n^2}{n!} \quad$ (involving only multiplications)

$$a_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

Ratio: $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)!} \, \frac{n!}{n^2} = \frac{(n+1)^2}{n+1} \, \frac{1}{n^2} = \frac{n+1}{n^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$$

By Ratio test, $\sum \frac{n^2}{n!}$ converges.

Root test

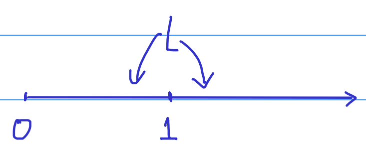
Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

- If $L < 1$: series $\sum a_n$ converges
- If $L > 1$: " " diverges
- If $L = 1$: test fails

Why is it true?

$\sqrt[n]{|a_n|} \approx L$ when n is large

$|a_n| \approx L^n$ " "



If $L < 1$ then $\sum L^n$ converges (Geometric series). Thus, $\sum a_n$ should converge.

If $L > 1$ then $|a_n| \approx L^n \to \infty$ as $n \to \infty$. Since $a_n \not\to 0$ as $n \to \infty$, the series diverges.

Ex: $\sum \frac{n^2}{2^n}$

$a_n = \frac{n^2}{2^n} > 0$ ..... $\sqrt[n]{a_n} = \frac{(\sqrt[n]{n})^2}{2}$

What is $\lim_{n\to\infty} \sqrt[n]{n}$?

In general, what is $\lim_{x\to\infty} \sqrt[x]{x}$? (x doesn't have to be integer)

$\sqrt[x]{x} = x^{\frac{1}{x}}$ ..... $\ln \sqrt[x]{x} = \frac{1}{x} \ln x$

$\lim_{x\to\infty} \ln \sqrt[x]{x} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=\!=} \lim_{x\to\infty} \frac{1/x}{1} = 0$

Exponentiate both sides: $\lim_{x\to\infty} \sqrt[x]{x} = 1$.

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$$

Return to the problem, $\lim \sqrt[n]{a_n} = \frac{1^2}{2} = \frac{1}{2} < 1$

By Root test, the series $\sum a_n$ converges.

* Recall:

If $\sum |a_n|$ converges then $\sum a_n$ also converges. The Converse is not true.

Series $\sum a_n$ is said to be:

- absolutely convergent if the series $\sum |a_n|$ converges, i.e.

$$\sum |a_n| < \infty$$

- conditionally convergent if $\sum a_n$ converges but $\underbrace{\sum |a_n| \text{ doesn't.}}_{\sum |a_n| = \infty}$

An abs. conv. series always converges. Any rearrangement of an abs. conv. series adds up to the same sum. In other words, an abs. conv. series is commutative. This is not the case for cond. conv. series.

Ex

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

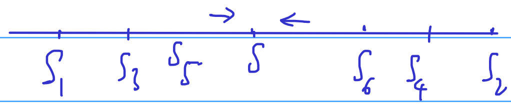
doesn't converge absolutely. why?

$$\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}} \quad \text{diverges}$$

(p-series with $p = 1/2 < 1$)

But the original series converges. why?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \overbrace{\underbrace{\overbrace{\underbrace{\overbrace{\underbrace{-\frac{1}{\sqrt{1}}}_{S_1} + \frac{1}{\sqrt{2}}}^{S_2} - \frac{1}{\sqrt{3}}}_{S_3} + \frac{1}{\sqrt{4}}}^{S_4} - \frac{1}{\sqrt{5}}}_{S_5} + \frac{1}{\sqrt{6}}}^{S_6} - \cdots = S$$



$$|S_{n+1} - S_n| = \frac{1}{\sqrt{n+1}} \to 0 \quad \text{as } n \to \infty$$

<u>Alternating series test</u>

$$\sum (-1)^n a_n$$

If $a_n \begin{cases} \geq 0 \\ \text{decreases } (a_{n+1} \leq a_n) \\ \text{goes to } 0 \end{cases}$ Then the series $\sum (-1)^n a_n$ converges.

<u>Ex</u>: $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$

$$a_n = \frac{1}{\sqrt{n}} \begin{cases} \geq 0 \\ \text{decreases} \\ \text{goes to } 0 \end{cases}$$

The series converges.

Note that Root test and Ratio test fail in this example.

$$\sqrt[n]{\left|\frac{(-1)^n}{\sqrt{n}}\right|} = \sqrt[n]{\frac{1}{\sqrt{n}}} = \frac{1}{(\sqrt[n]{n})^{1/2}} \longrightarrow 1 \quad \text{as } n \to \infty$$

<u>Power series</u>

$$\sum \underbrace{a_n x^n}_{\text{only multiplications involved}} \quad \text{where } x \text{ is a variable.}$$

Ratio test and Root test are useful.

Continue next time.