Lecture 23 (3/8/2019)

More examples on Alternating Series test:

$$\sum (-1)^{n+1} n e^{-n} = -\sum (-1)^n \underbrace{n e^{-n}}_{a_n}$$

Need:

$$a_n \begin{cases} \geq 0 \\ \text{decreases (for } n \text{ large)} \\ \text{goes to } 0 \end{cases}$$

$$a_n = \frac{n}{e^n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$f(x) = x e^{-x}$

$f'(x) = (1-x) e^{-x} < 0$ if $x > 1$

f is decreasing on $(1, \infty)$.

Thus, a_n is decreasing when $n \geq 2$.

The series converges.

Power series

$\sum a_n x^n$ ------ power series in x about 0

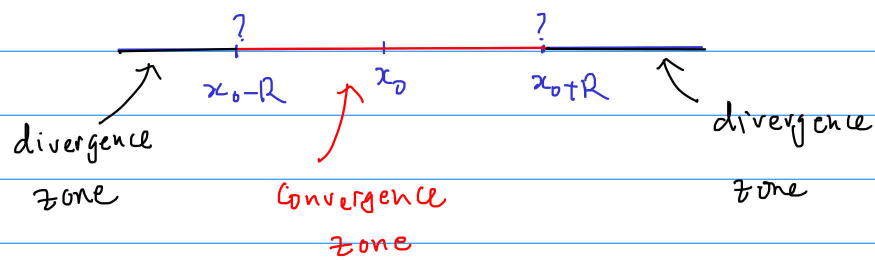
$\sum a_n (x - x_0)^n$ ----- " " " x_0

x: variable

a_n: coefficient of x^n

x_0: base point

The convergence/divergence of this series depends on x.



There is $R \in [0, \infty]$ such that the series

$$\begin{cases} \text{abs. conv. if} & |x - x_0| < R, \\ \text{diverges if} & |x - x_0| > R. \end{cases}$$

If $R = 0$: $x = x_0$ is the only value of x for which the series converges.

If $R = \infty$: the series conv. for any value of x.

R is called the radius of convergence of the series.

How to find the radius of conv.? Use Ratio test or Root test

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

$a_n = \dfrac{1}{n}$

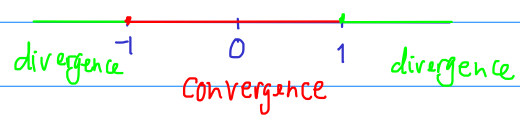
General term $b_n = \dfrac{x^n}{n}$.

$$\frac{b_{n+1}}{b_n} = \frac{x^{n+1}}{n+1} \frac{n}{x^n} = x \frac{n}{n+1} \xrightarrow{n \to \infty} x$$

$$\left| \frac{b_{n+1}}{b_n} \right| \xrightarrow{n \to \infty} |x|$$

Ratio test $\begin{cases} \text{If } |x| < 1 : & \sum \frac{x^n}{n} \text{ converges (absolutely)} \\ \text{If } |x| > 1 : & \quad \text{"} \quad \text{diverges.} \end{cases}$

Radius of convergence is $R = 1$.



At $x = 1$: $\sum \frac{1}{n}$ diverges $(= \infty)$ (harmonic series)

At $x = -1$: $\displaystyle\sum \frac{(-1)^n}{n}$ converges (alternating series test)

Interval of convergence is $[-1, 1)$.

Ex: $\displaystyle\sum_{n=0}^{\infty} x^n$ $\quad (a_n = 1)$

General term $b_n = x^n$

$$|b_n|^{1/n} = |x^n|^{1/n} = |x|$$

Root test $\begin{cases} \text{If } |x| < 1 \text{ then series converges absolutely.} \\ \text{If } |x| > 1 \quad \text{"} \quad \text{diverges.} \end{cases}$

At $x = 1$: $\displaystyle\sum_{n=0}^{\infty} 1 = \infty$ (diverges)

At $x = -1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ diverges because $\displaystyle\lim_{n\to\infty} (-1)^n \neq 0$.

Interval of convergence is $(-1, 1)$.

Ex: $\displaystyle\sum \frac{(-1)^n}{n^2} (x-1)^n$

Radius of convergence $R = 1$

Interval of convergence: $[0, 2]$