Lecture 24 (31/1/2019)

More examples on power series:

Ex: $$\sum \frac{(x-2)^n}{3^n n}$$ ----- Power series with base point 2.

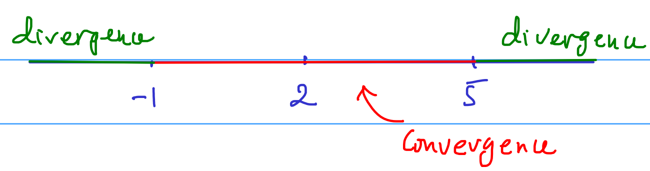
General term: $$b_n = \frac{(x-2)^n}{3^n n}$$

Root test: $$|b_n| = \frac{|x-2|^n}{3^n n}, \quad |b_n|^{1/n} = \frac{|x-2|}{3\sqrt[n]{n}}$$

$$\lim |b_n|^{1/n} = \frac{|x-2|}{3}$$

If $\frac{|x-2|}{3} < 1$ then series converges.

If $\frac{|x-2|}{3} > 1$ " " diverges.



Radius of conv. $R = 3$

The borderline points -1 and 5 are examined separately.

Interval of conv: $[-1, 5)$

Ex: $$\sum n^n x^n$$

$$|b_n|^{1/n} = n|x| \to \infty \quad \text{for any } x \neq 0$$

Series diverges for any $x \neq 0$.

Radius of conv. $= 0$.

Interval of conv. $= [0, 0] = \{0\}$.

Ex: $$\sum \frac{x^n}{n!}$$

$$\left.\begin{aligned} b_n &= \frac{x^n}{n!} \\ b_{n+1} &= \frac{x^{n+1}}{(n+1)!} \end{aligned}\right\} \quad \left|\frac{b_{n+1}}{b_n}\right| = \frac{x}{n+1} \xrightarrow{n\to\infty} 0 < 1$$

The series converges absolutely for all x.

Radius of conv. $R = \infty$

Interval of conv. $(-\infty, \infty) = \mathbb{R}$

<u>Taylor series</u>

Given a (smooth) function $f(x)$. Can one express f as a power series?

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$= \underbrace{a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + a_4(x-x_0)^4 + \dots}$$

decomposition of $f(x)$ into simple "modes", each is a nomial.

Such expression (decomposition) has a number of advantages, including:

- Approximate a function by polynomials, facilitating numerical calculation.

$$\int_a^b f(x)\,dx \text{ "="} a_0 \int_a^b 1\,dx + a_1 \int_a^b (x-x_0)\,dx + a_2 \int_a^b (x-x_0)^2\,dx$$

$$+ a_3 \int_a^b (x-x_0)^3\,dx + \dots$$

More applications will be discussed next time.

* Starting with a function f, how to find a series representation of f?

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

How to compute $a_0, a_1, a_2, a_3, \dots$?

Plug $x=0$: $f(0) = a_0$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

Plug $x=0$: $f'(0) = a_1$

$$f''(x) = 2a_2 + 3(2)x + 4(3)x^2 + \dots$$

Plug $x=0$: $f''(0) = 2a_2$, thus $a_2 = \frac{1}{2} f''(0)$

......

$$a_n = \frac{f^{(n)}(0)}{n!}$$

In general, the only power series that could possibly represent f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

This is called the Taylor series of f about point x_0.

Question: is $f(x)$ actually equal to its Taylor series?

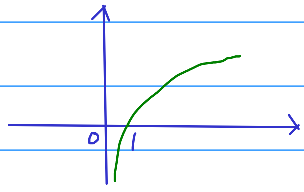
Answer: yes for most functions. This type of functions is called analytic functions.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

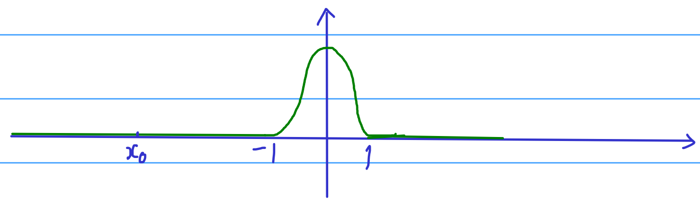
$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$



Logarithm function is not defined at $x=0$.

Analytic functions have an interesting feature: to know f function at all points x, it suffices to know f and all its derivatives at only one point.

Example of non-analytic functions:



$$f(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } -1 < x < 1 \\ 0 & \text{if } x \leq -1 \text{ or } x \geq 1 \end{cases}$$

If $x_0 < -1$ then $f^{(n)}(x_0) = 0$ for all $n \geq 1$.

Taylor series of f: $f(0) + \underbrace{f'(0)}_{0} x + \underbrace{f''(0)}_{0} x^2 + \cdots = f(0) = e^{-1}$

which is a constant function.

But f itself is not a constant function.