Lecture 25 (3/13/2019)

Power series $\sum a_n x^n$ defines a function on its interval of convergence.

$$f(x) := \sum_{n=0}^{\infty} a_n x^n$$

Let R be the radius of convergence. We know that on the interval $(-R, R)$ the series converges absolutely.

On $(-R, R)$, f is a nice, smooth function:

- differentiable at any order,
- integrable,
- analytic (by the fact that f is equal to a power series)

  $\sum a_n x^n$ is exactly the Taylor series of f (about 0).

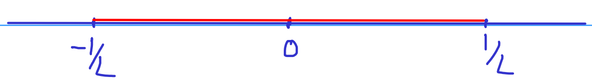
Recall: how to find the radius of convergence of $\sum a_n x^n$?

Suppose $\lim |a_n|^{1/n} = L$.

Then $\lim |a_n x^n|^{1/n} = L|x|$

If $L|x| < 1$ then the series converges absolutely.

If $L|x| > 1$ then the series diverges.



The radius of convergence is $R = 1/L$, with the understanding that

$R = \infty$ if $L = 0$,

$R = 0$ if $L = \infty$.

Now consider the "derivative series":

$$\underbrace{\sum_{n=0}^{\infty} (a_n x^n)'}_{a_0' + (a_1 x)' + (a_2 x^2)' + (a_3 x^3)' + \dots} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$= 0 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$

We claim that the derivative series has the same radius of convergence as the original series. Why?

$$\sum a_n n x^{n-1} = \sum \underbrace{\frac{a_n n}{x}}_{b_n} x^n$$

Root Test: $\lim\limits_{n\to\infty} |b_n|^{1/n} = \lim \frac{|a_n|^{1/n} n^{1/n}}{|x|^{1/n}} = \lim |a_n|^{1/n} = L$

Thus, the radius of convergence of the derivative series is also $R = \frac{1}{L}$.

* Conclusion: on the interval $(-R, R)$, differentiation and integration of the power series can be taken term by term.

$$f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n\,dx$$

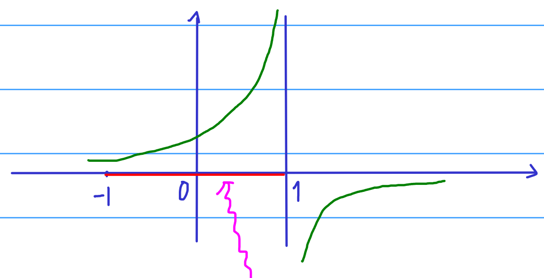
Ex:

$$\underbrace{1 + q + q^2 + q^3 + \dots}_{\text{geometric series}} = \frac{1}{1-q} \quad \text{for all } |q| < 1$$

Replace q by x: $\frac{1}{1-x} = \underbrace{1 + x + x^2 + x^3 + \dots}_{\text{This is the Taylor series (about 0) of function } \frac{1}{1-x}}$

Recall that Taylor series is the only power series that can represent a given function f.

The radius of convergence is $R=1$ (easy to check)



Take derivative : $$\frac{1}{(1-x)^2} = 0+1+2x+3x^2+4x^3+\cdots$$

We have found Taylor series of function $\frac{1}{(1-x)^2}$ quite easily.

Recall that power series of a function f about x is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Normally one would need to differentiate f, then plug $x=0$,
differentiate again, then plug $x=0$,
....

But sometimes this whole process can be replaced by purely algebraic operations.

Ex :

Find Taylor series about 0 of $f(x) = \frac{x}{1-x^2}$.

$$\frac{x}{1-x^2} = x\frac{1}{1-x^2} = x\left(1+x^2+x^4+x^6+x^8+\cdots\right)$$
$$= x+x^3+x^5+x^7+x^9+\cdots$$

Ex :

Find Taylor series about 0 of $f(x) = \frac{x+x^3}{1-x^2}$.

$$\frac{x+x^3}{1-x^2} = (x+x^3)\frac{1}{1-x^2} = (x+x^3)(1+x^2+x^4+x^6+x^8+\cdots)$$

$$= 0 + x(1) + x^2(0) + x^3(1+1) + x^4(0) + x^5(1+1) + \cdots$$

$$= x + 2x^3 + 2x^5 + 2x^7 + 2x^9 + \cdots$$

Ex:

Find Taylor series about 0 of $f(x) = e^x \sin x$

$$e^x \sin x = \left(1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= 0 + x(1) + x^2(1) + x^3\left(-\frac{1}{6} + \frac{1}{2}\right) + x^4\left(-\frac{1}{6} + \frac{1}{6}\right) + \cdots$$

(by distribution)