Lecture 26 (3/15/2019)

* More examples of Taylor series:

Ex: What is Taylor series of $\arctan x$ about 0?

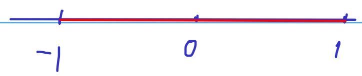
Observe: $$(\arctan x)' = \frac{1}{1+x^2}$$

Recall: $$\frac{1}{1-t} = 1+t+t^2+t^3+\cdots \qquad (-1<t<1)$$

Substitute $t=-x^2$:

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-\cdots \qquad (-1<x<1)$$

Thus, $$(\arctan x)' = 1-x^2+x^4-x^6+x^8-\cdots$$



On $(-1,1)$, the series converges absolutely. Integration of the whole series amounts to integration term by term.

$$\int_0^t (\arctan x)'\,dx = \int_0^t \left(1-x^2+x^4-x^6+x^8-\cdots\right)dx$$

$$= x\Big|_0^t - \frac{x^3}{3}\Big|_0^t + \frac{x^5}{5}\Big|_0^t - \frac{x^7}{7}\Big|_0^t + \frac{x^9}{9}\Big|_0^t - \cdots$$

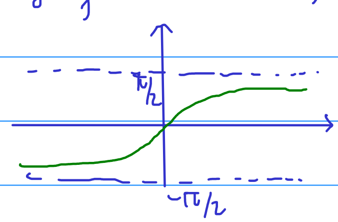
$$\arctan t + C = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \cdots$$

Plug $t=0$: $0+C=0$, so $C=0$.

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \cdots$$

Note:

the equality is valid only for $t\in(-1,1)$, although arctan is a smooth function on $\mathbb{R}$.



Recall: • If a function is defined as a power series, then it is nice and smooth inside the interval of conv. (the borderline points excluded).

$$f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{smooth on } (x_0 - R, x_0 + R)$$

• If one starts with a smooth function on some interval $(-\alpha, \alpha)$, it might not be represented by a power series on $(-\alpha, \alpha)$. Example is seen above.

Ex: If f is even, i.e. $f(x) = f(-x)$ then the Taylor series of f (about 0) only contains even powers.

why?

same function, two power series

$$\begin{cases} f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ f(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - \cdots \end{cases}$$

$\rightarrow$ They must be the same: $a_1 = a_3 = a_5 = a_7 = \cdots = 0$

This is why $\underbrace{\cos x}_{\cos x = \cos(-x)}$ has only even powers in its series.

Ex:

If f is odd, i.e. $f(x) = -f(-x)$, then there are only odd powers in the Taylor series of f.

* Application to differential equations:

Ex: $y' - xy = e^x$, $y(0) = 1$

One can solve this eq. by integration-factor method. Here we view this eq. in the light of Taylor series. To find y is more or less equivalent to finding its Taylor series (about 0).

$$
\begin{array}{rcl}
y = y(x) &=& a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
y' &=& a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\
- \quad xy &=& a_0 x + a_1 x^2 + a_2 x^3 + \cdots \\
\hline
e^x &=& a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x^2 + (4a_4 - a_2)x^3 + \cdots
\end{array}
$$

We know that the power series representing e^x is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The above two series must be the same:

$$
\begin{cases}
a_1 = 1 & \\
2a_2 - a_0 = 1 & (\text{note that } a_0 = 1 \text{ by } y(0) = 1) \rightsquigarrow a_2 = 1 \\
3a_3 - a_1 = 1/2 & \rightsquigarrow a_3 = 1/2 \\
4a_4 - a_2 = 1/6 & \rightsquigarrow a_4 = 1/6 \\
\cdots & \cdots
\end{cases}
$$

Ex: $\quad y' = \sin(xy), \quad y(0) = 1$

It's not clear how to solve for y analytically because the equation is nonlinear. But one can attempt to solve for y approximately by finding its Taylor polynomials.

$$y = y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$\sin t = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots$$

Plug $t = xy$:

$$\sin(xy) = xy - \frac{(xy)^3}{6} + \frac{(xy)^5}{120} - \cdots$$

$$= x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) - \frac{1}{6}x^3(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^3 + \underbrace{h.o.t}_{\text{higher order terms}}$$

$$\text{RHS} = x a_0 + x^2 a_1 + x^3\left(a_2 - \tfrac{1}{6}a_0^3\right) + h.o.t$$

$$\text{LHS} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

The two series must be the same:

$$\begin{cases} a_1 = 0 \\ 2a_2 = a_0 \quad (\text{note: } a_0 = 1 \text{ by } y(0) = 1) \rightsquigarrow a_2 = 1/2 \\ 3a_3 = a_1 \rightsquigarrow a_3 = 0 \\ 4a_4 = a_2 - \tfrac{1}{6}a_0^3 \rightsquigarrow a_4 = \tfrac{1}{2} - \tfrac{1}{6}1^3 = \tfrac{1}{3} \\ \dots \qquad \dots \end{cases}$$

* Topics to be covered in final exam

- Eigenvalues / eigenvectors ----- especially involving complex numbers
- $\underbrace{\text{Taylor polynomials}}_{\text{truncation of Taylor series}}$ & Taylor series
- Convergence tests:
  - Comparison test
  - "$a_n \to 0$" test
  - Integral test
  - Alternating series test
  - Ratio test
  - Root test