Lecture 3 (1/11/2019)

Linear maps:

$\underbrace{f(x+y)}_{\substack{\text{add first,} \\ \text{then apply } f}}$ $\qquad$ $\underbrace{f(x)+f(y)}_{\substack{\text{apply } f \text{ first,} \\ \text{then add}}}$

The results are generally different.

For f to be linear, the results must always be equal.

$\underbrace{f(cx)}_{\substack{\text{scale first,} \\ \text{then apply } f}}$ $\qquad$ $\underbrace{cf(x)}_{\substack{\text{apply } f \text{ first,} \\ \text{then scale}}}$

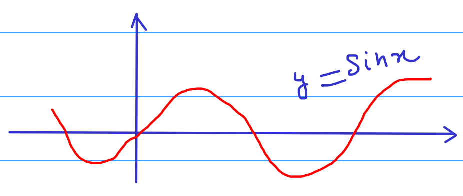
Most maps are nonlinear.

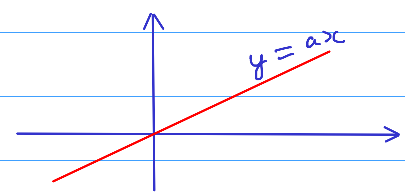
Ex:

$$f(x) = \sin x$$

There are several ways to see that f is nonlinear.

① Any linear map from $\mathbb{R}$ to $\mathbb{R}$ must be of the form $f(x) = ax$ where a is a constant.





The sine function is not of this form.

② Take $x = y = \frac{\pi}{2}$:

Add first: $x + y = \pi$. Then apply f: $\sin(\pi) = 0$.

Apply first: $\sin(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. Then add: $1 + 1 = 2$

The results are different. Thus, f is <u>not</u> additive.

③ Take $x = \frac{\pi}{2}$, $c = 2$:

Scale first: $2 \times \frac{\pi}{2} = \pi$. Then apply f: $\sin(\pi) = 0$

Apply f first: $\sin(\frac{\pi}{2}) = 1$. Then scale: $2 \times 1 = 2$

The results are different. Thus, f is <u>not</u> scalar-multiplicative

* Recall:

$$\text{linear maps} \begin{cases} \mathbb{R} \to \mathbb{R} \quad \cdots\cdots \quad f(x) = ax \\ \mathbb{R}^2 \to \mathbb{R} \quad \cdots\cdots \quad f(x_1, x_2) = a_1x_1 + a_2x_2 \quad (1) \\ \mathbb{R}^2 \to \mathbb{R}^2 \quad \cdots\cdots \quad f(x_1, x_2) = (a_1x_1 + a_2x_2, a_3x_1 + a_4x_2) \quad (2) \end{cases}$$

(1): $f(x) = \underbrace{(a_1, a_2)}_{a} \cdot \underbrace{(x_1, x_2)}_{x} = a \cdot x$

(2): $f(x) = \underbrace{A}_{\text{matrix}} \cdot x$

the multiplication needs to be defined.

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

* Matrix: is introduced to represent linear maps.

An $m \times n$ matrix is an $m \times n$ table of numbers, called <u>entries</u> or (coefficients)

<u>Ex</u>: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad 2 \times 3$ matrix: $\begin{cases} 2 \text{ rows} \\ 3 \text{ columns} \end{cases}$

entry at position $(1,2)$: 1^{st} row, 2^{nd} col.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 3 \\ 4 & 5 & -2 \end{bmatrix}$$

3×3 matrix, or square matrix of size 3

diagonal

Matrix A of size $m \times n$. Denote by a_{ij} the entry at position (i,j), i.e. i'th row and j'th column.

$[1 \quad 2 \quad -4]$ 1×3 matrix, called <u>row vector</u>

$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ 3×1 matrix, called <u>column vector</u>

$[3]$ 1×1 matrix, number, or scalar

* Algebra on matrices:

- $A+B$ ..... valid only if A and B are of the same size.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -3 & 0 \\ 5 & -6 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 9 & -1 & 13 \end{bmatrix}$$

addition is performed component-wise

- scalar multiplication (or scaling): cA

$$2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

scaling is performed component-wise

$0A = \underset{\sim}{0}$

zero matrix (all entries equal to 0)

- transpose: A^T

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The rows of A become the columns of A^T. The cols of A become the rows of A^T.

If A is $m\times n$ matrix then A^T is $n\times m$ matrix.

- Multiplication: AB

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & -2 \\ 0 & 12 \end{bmatrix}$$

This is not the type of multiplication we are talking about.

We want a type of multiplication that reflects the composition of two linear maps. Not any two linear maps can be composed with each other. For example,

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad g : \mathbb{R}^2 \to \mathbb{R}^3$$

$f \circ g$ doesn't exist

Not any two matrices can be multiplied with each other. Their sizes must be "compatible".

The # of cols of A must be equal to # rows of B

If $\begin{cases} A \text{ ---- } m \times n \\ B \text{ ---- } n \times p \end{cases}$ then AB is $m \times p$ matrix.

Ex:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1 \times 3} \underbrace{\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}}_{3 \times 1} = 1 \times (-1) + 2 \times 3 + 3 \times 4 = 17$$

row vector $\times$ col. vector $=$ number (1×1 matrix)

same length

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{3 \times 3} \underbrace{\begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -2 \end{bmatrix}}_{3 \times 2} = \underbrace{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}_{3 \times 2}$$

entry at position $(1,2)$

$=$ 1^{st} row of A $\times$ 2^{nd} col. of B

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 1 \times 2 + 2 \times 0 + 3 \times (-2) = -4$$

entry at pos. $(3,2)$

$=$ 3^{rd} row of A $\times$ 2^{nd} col. of B

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = -2$$