Lecture 4 (1/14/2019)

$*$ Matrix:

$A+B$ ...... A, B must be of the same size (same # rows, same # cols)

cA ..... any $c \in \mathbb{R}$, any matrix A

Combine: $A - B = A + (-1)B$

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 4 & 4 \end{bmatrix}$$

. Transpose:

A .... $m \times n$

A^T ..... $n \times m$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

rows of A are cols. of A^T

cols. of A are rows of A^T

. Multiplication:

$$\underset{m \times n}{A}\,\underset{n \times p}{B} \;\ldots\ldots\; m \times p$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1 \times 3} \underbrace{\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}}_{1 \times 3}$$ ..... not defined, sizes incompatible

$$\underbrace{\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}}_{3 \times 1} \underbrace{\begin{bmatrix} 5 & -3 & 7 \end{bmatrix}}_{1 \times 3} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{3 \times 3}$$

position (2,3): 2^{nd} row $\cdot$ 3^{rd} col $= 0 \times 7 = 7$

* Linear map: $f: \mathbb{R}^n \to \mathbb{R}^m$

want: $f(x) = Ax$

- Common practice:

View a vector in $\mathbb{R}^n$ as an $n \times 1$ matrix (column vector).

Ex:

$x = (2, 0, 3)$ ..... $x = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

- Know: a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the form

$f(x_1, x_2) = (a_1 x_1 + a_2 x_2, \; a_3 x_1 + a_4 x_2)$

$\Big\updownarrow$ matrix

$$\underbrace{f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}_{x} = \begin{bmatrix} a_1 x_1 + a_2 x_2 \\ a_3 x_1 + a_4 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

$$f(x) = Ax$$

matrix

column vector

matrix multiplication

- Ex:

linear maps from $\mathbb{R}^2$ to $\mathbb{R}$ has the form

$f(x_1, x_2) = a x_1 + b x_2$

$\Big\updownarrow$ matrix

$$\underbrace{f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}_{f(x)} = [a x_1 + b x_2] = \underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

- In general, $f: \mathbb{R}^n \to \mathbb{R}^m$ linear

  In matrix form, $f(x) = Ax$

matrix representing f | associated with f

$m \times 1$ $\quad$ $m \times n$ $\quad$ $n \times 1$

How to find A? two ways — list rows
— list cols.

<u>Ex</u>:

$f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_2 + x_3,\ x_1 - x_2 + x_3)$

A ... 2×3 matrix

- Find A by listing its rows:

$[0 \quad 1 \quad 1]$ $\qquad$ $[1 \quad -1 \quad 1]$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Look at each output of f. List the factor in front of each variable (make sure in the right order).

- Find A by listing its cols:

$$\left.\begin{aligned} e_1 &= (1,0,0) \\ e_2 &= (0,1,0) \\ e_3 &= (0,0,1) \end{aligned}\right\} \text{ standard basis of } \mathbb{R}^3$$

$$f(e_1) = f(1,0,0) = (0,1)$$
$$f(e_2) = f(0,1,0) = (1,-1)$$
$$f(e_3) = f(0,0,1) = (1,1)$$

$$A = \begin{bmatrix} | & | & | \\ f(e_1) & f(e_2) & f(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

In general, $f: \mathbb{R}^n \to \mathbb{R}^m$

standard basis $e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

$\ldots$

$e_n = (0, 0 \ldots, 0, 1)$

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix}}_{n \text{ cols.}} \Big\} \, m \text{ rows}$$

* Algebra of linear maps:

$$\left.\begin{array}{l} f: \mathbb{R}^n \to \mathbb{R}^m \\ g: \mathbb{R}^n \to \mathbb{R}^m \end{array}\right\} \; f+g \text{ ---- is a map defined by}$$

$$(f+g)(x) = f(x) + g(x)$$

Ex:

$f(x) = \sin x$

$g(x) = e^x$

$f+g$ is a map that takes x to $\sin x + e^x$.

f ------ A

g ------ B

$f+g$ ----- $A+B$

• Scaling:

cf ---- the map that takes x to $cf(x)$

Ex:

$f(x) = \sin x$

$2f$ is the map that takes x to $2 \sin x$

$f \cdots\cdots A$

$cf \cdots\cdots cA$

- Composition:

$$\mathbb{R}^p \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$$

$$x \longmapsto g(x) \longmapsto f(g(x))$$

Composite map, denoted by $f \circ g$

$f \circ g$: g takes argument, then f

Ex:

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$$

$$g(x_1, x_2) = (x_1 + x_2, x_1)$$

$$f(x_1, x_2) = (0, x_1 - x_2, x_2)$$

What is $f \circ g$?

$$f \circ g\,(x_1, x_2) = f(g(x_1, x_2)) = f(x_1 + x_2, x_1)$$

$$f(y_1, y_2) = (0, y_1 - y_2, y_2)$$

Here $y_1 = x_1 + x_2$, $y_2 = x_1$

$$= (0, x_1 + x_2 - x_1, x_1) = (0, x_2, x_1)$$

$$f \cdots A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$g \cdots B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$f \circ g \cdots C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = AB$$

Matrix multiplication is defined (the way we know) so that it represent the composition of linear maps.