Lecture 5 (1/16/2019)

Review:

$$f: \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x_1, x_2) = (0, x_1, x_1 - x_2)$$

$$g: \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x_1, x_2) = (x_1 + x_2, x_1 + 2x_2)$$

$$f \cdots\cdot A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \underbrace{1}_{f(e_1)} & \underbrace{-1}_{f(e_2)} \end{bmatrix}$$

$$g \cdots\cdots B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^3 \end{tikzcd}$$

$f \circ g \cdots\cdot$ g takes input first, the output of g becomes the input of f.

$$f \circ g\,(x_1, x_2) = f(g(x_1, x_2))$$

$$= f(x_1 + x_2, x_1 + 2x_2)$$

Rewrite the def. of f, using different variables:

$$f(y_1, y_2) = (0, y_1, y_1 - y_2)$$

$$f(\underbrace{x_1 + x_2}_{y_1}, \underbrace{x_1 + 2x_2}_{y_2}) = (0, x_1 + x_2, \underbrace{x_1 + x_2 - (x_1 + 2x_2)}_{-x_2})$$

$$f \circ g\,(x_1, x_2) = (0, x_1 + x_2, -x_2)$$

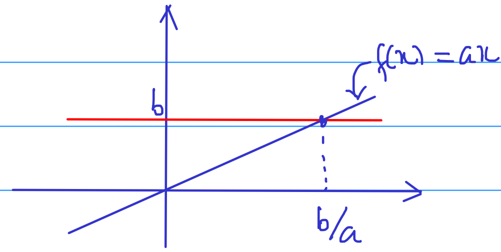
$$\updownarrow$$

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} = AB$$

* $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear.

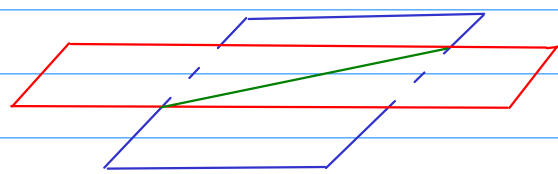
Given $b \in \mathbb{R}^m$. How to solve the equation $f(x) = b$?

$f: \mathbb{R} \to \mathbb{R}$



$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$



the equation $f(x_1, x_2) = b$ is the equation of intercepts of the graph of f (a plane) and the plane $x_3 = b$.

In this case: 2 possibilities
- no solutions (two planes parallel)
- infinitely many solutions

Geometric approach is not efficient if m and n are large.

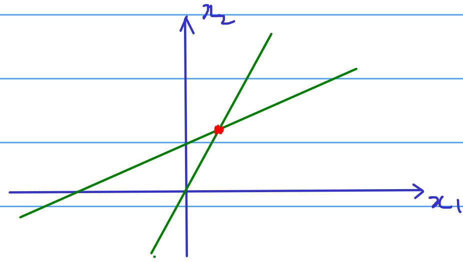
Algebraic approach:

$$f: \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (2x_1 - x_2, x_1 + 2x_2)$$

$$b = (0, 5)$$

The equation $f(x) = b$ becomes a system of equations

$$(\mathrm{I}): \begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 2x_2 = 5 \end{cases}$$



Matrix form: $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

Augmented matrix associated with the system (I) is

$$\underbrace{\begin{matrix} 2 & -1 \\ 1 & 2 \end{matrix}}_{A} \; \Big| \; \underbrace{\begin{matrix} 0 \\ 5 \end{matrix}}_{b}$$

Strategy: eliminate x from the second equation, for example by subtracting $\frac{1}{2}$ times the first eq. from the second equation.

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 5/2 & 5 \end{array}\right]$$

Then do back substitution:

2nd row: $\frac{5}{2} x_2 = 5 \Rightarrow x_2 = 2$

1st row: $2x_1 - x_2 = 0 \Rightarrow 2x_1 - 2 = 0 \Rightarrow x_1 = 1$

Three row operations: (called elementary row operations)

- Multiply a row by a nonzero number
- Exchange the order of two rows.
- Replace a row by the sum of itself and a multiple of another row.

Goal: use elementary row operations to bring the augmented matrix into echelon form.

This process is called <u>Gauss elimination</u>.

Ex:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \cdots \text{augmented matrix of the system}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 \quad\quad - x_3 = 3 \end{cases}$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 + 6R_2]{R_2 = R_2/-5} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

Then do back substitution:

3rd row: $-4x_3 = -12 \Rightarrow x_3 = 3$

2nd row: $x_2 + x_3 = 2 \Rightarrow x_2 = 2 - x_3 = 2 - 3 = -1$

1st row: $x_1 + 2x_2 + 3x_3 = 9 \Rightarrow x_1 = 9 - 2x_2 - 3x_3 = 9 - 2(-1) - 3(3)$

$= 2$