Lecture 6 (1/18/2019)

System of linear equations:

$$\begin{cases} 2x + y = -2 \\ 3x + 2y = 4 \end{cases}$$

Matrix form:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}_{b}$$

Augmented matrix:

$$[A \mid b] = \left[\begin{array}{cc|c} 2 & 1 & -2 \\ 3 & 2 & 4 \end{array}\right]$$ encodes all information needed to solve the system

We want to eliminate x form the second equation. This is done by several ways: one can subtract $3/2$ times the first row from the second row, which gives

$$\left[\begin{array}{cc|c} 2 & 1 & -2 \\ 0 & 1/2 & 7 \end{array}\right]$$

Or one can divide the first row by 2, then subtract 3 times the first row from the second row. This method gives

$$\left[\begin{array}{cc|c} 1 & 1/2 & -1 \\ 0 & 1/2 & 7 \end{array}\right]$$

Both systems are equivalent to the original one, and equivalent to each other.

Then we do back substitution:

$$\begin{array}{c} \begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & -2 \\ 0 & 1/2 & 7 \end{array}\right] \uparrow \end{array}$$

The second eq. gives $\frac{1}{2}y = 7$. Thus $y = 14$.

The first eq. gives $2x + y = -2$. Thus, $x = \dfrac{-2-y}{2} = -8$

* Elementary row operations:

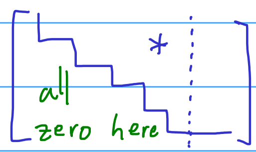
1) Multiply a row by a nonzero number
2) Interchange two rows.
3) Replace a row by the sum of itself an a multiple of another row.

Any of these operations makes sure that the new system is equivalent to the old one.

*Strategy:

This is called Gauss elimination method

- Write the augmented matrix of the system.
- Use row operations consecutively to bring the original augmented matrix into row echelon form:



- Back substitution from bottom to top.

Definition:

The first nonzero entry of a row (counting from the left) is called a pivot entry.

A matrix is in row echelon form if:

- for any two consecutive rows, the pivot entry of the row above is further to the left of the pivot entry of the row below.
- all zero rows are gathered at the bottom.

Examples provided on another sheet (check course website).

Ex:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 \quad - x_3 = 3 \end{cases}$$

Augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 + 6R_2]{R_2 = R_2/-5} \left.\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]\right\} \text{row echelon form}$$

Then do back substitution:

3^{rd} row: $-4x_3 = -12 \Rightarrow x_3 = 3$

2^{nd} row: $x_2 + x_3 = 2 \Rightarrow x_2 = 2 - x_3 = 2 - 3 = -1$

1^{st} row: $x_1 + 2x_2 + 3x_3 = 9 \Rightarrow x_1 = 9 - 2x_2 - 3x_3 = 9 - 2(-1) - 3(3)$
$= 2$

Ex:

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 \quad = 2 \\ 4x_1 - x_2 + 2x_3 = 3 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 4 & -1 & 2 & 3 \end{array}\right] \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -2 & -1 \end{array}\right]$$

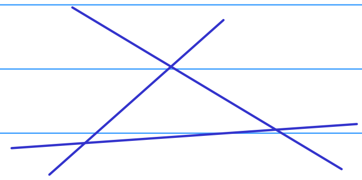
$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$ row echelon form

Back substitution: the last row implies $0 = -1$ (!)

The system is inconsistent (having no solutions).

The system is inconsistent if there appears a row $[\,0 \;\; 0 \; \ldots \; 0 \mid a\,]$.

nonzero



3 planes have no common points.