Lecture 7 (1/23/2019)

* System with infinitely many solutions:

Ex:

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 - x_3 = 1 \\ 4x_1 - x_2 + x_3 = 1 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 4 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

row echelon form

this column has no pivot entry

The columns without pivot entries give us a free variable.

In this case, x_3 is a free variable and there are infinitely many solutions.

$x_3 = t$

Back substitution:

3rd row : no useful information

2nd row : $3x_2 - 3t = 1 \Rightarrow x_2 = \frac{1+3t}{3} = \frac{1}{3} + t$

1st row : $x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 - x_3 = \frac{1}{3}$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ \frac{1}{3}+t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 \\ 1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

Ex: A system of 3 equations and 4 unknowns has augmented matrix [A|b]. Suppose that after several row reduction steps, one gets

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

How many solutions are there?

Answer: no solutions. Although the 3rd and 4th cols are without pivot entries, the system is inconsistent due to the last row.

Ex:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & & \\ x_1 & x_2 & x_3 & x_4 & & \end{array} \end{array}$$

How many solutions are there?

Answer: Infinitely many.

The 3rd and 4th cols are without pivot entries. Thus, x_3 and x_4 are free variables.

$x_3 = s$, $x_4 = t$

Back substitution:

3rd row: no useful information

2nd row: $2x_2 - x_3 = 2 \Rightarrow x_2 = \frac{2+x_3}{2} = 1 + s/2$.

1st row: $x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \Rightarrow x_1 = -2x_2 - 3x_3 - 4x_4$

$$= -2 - s - 3s - 4t$$

$$= -2 - 4s - 4t$$

Conclusion:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2-4s-4t \\ 1+s/2 \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

System of linear equations
- no solutions ------ inconsistent
- 1 sol.
- infinitely many

} consistent

These are all possible scenarios.

no sols: when there occurs a row $[\,0\ 0 \cdots 0 \mid a\,]$

$\uparrow$ nonzero

inf. many sols: when there are no such rows and there is at least one col. without pivot entries.

* Reduced row echelon form of a matrix (RREF):

RREF is REF (row echelon form) with two additional properties:

- All pivot entries must be 1 (called pivot 1)
- If a column contains a pivot 1, all other entries on that column must be 0.

A column containing a pivot 1 is said to be a <u>pivot column</u>.

<u>Ex</u>: See the example sheet posted on course website.