Lecture 8 (1/25/2019)

1 is the identity element of the multiplication of numbers:

$$a \cdot 1 = 1 \cdot a = a \quad \text{for all } a \in \mathbb{R} \qquad (*)$$

It is the unique number having property (*).

How about the multiplication of matrice? Is there an identity element?

$$AI = IA = A \quad \text{for all matrix } A \text{ of size } n \times n.$$

Look at the linear map picture:

A represents linear map $f: \mathbb{R}^n \to \mathbb{R}^n$

I represents linear map $g: \mathbb{R}^n \to \mathbb{R}^n$

AI 〃 $f \circ g$

IA 〃 $g \circ f$

The question becomes: is there a linear map $g: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f \circ g = g \circ f = f \quad \text{for all linear maps } f: \mathbb{R}^n \to \mathbb{R}^n \,?$$

Yes, $g = id$ (the identity map on $\mathbb{R}^n$) is such a map.

$id(x) = x$, or

$id(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$

This map is associated with matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \leftarrow \text{called identity matrix}$$

Ex:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_1 = 1$$

*Property:

$$AI_n = I_n A = A \quad \text{for all } n \times n \text{ matrix } A.$$

(Let A and B be two $n \times n$ matrices. In general, $AB \neq BA$. But one always have $AI_n = I_n A$)

I_n plays a role as number 1 in multiplication.

Def:

Let A be an $n \times n$ matrix. If B is an $n \times n$ matrix such that

$$AB = BA = I_n$$

then A is said to be invertible and B is called the inverse of A, denoted by $B = A^{-1}$

Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible

because for any 2×2 matrix B, the product AB is always $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible

because $BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ is invertible because

$$A\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

How to check if a matrix is invertible, and if so, how to find the inverse?

Given $n \times n$ matrix A, we follow the steps:

- Form a matrix $[A \mid I_n]$ (an $n \times 2n$ matrix, with a vertical bar in the middle)

- Use elementary row operations to change this matrix into row echelon form (REF)

Two scenarios:

- If all columns on the left of the bar contain pivot entry then A is invertible. Continue to use row operations to change the $n \times 2n$ matrix into RREF.

$$[A \mid I_n] \longrightarrow \cdots \longrightarrow [I_n \mid \underbrace{A^{-1}}_{\text{inverse of } A}]$$

- If a column on the left of the bar contains no pivot entries, then stop. A is not invertible.

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -4 & -3 & 1 \end{array}\right]$$

REF, and good

A is invertible! Continue:

$$\xrightarrow[R_1 = R_1 - 2R_2]{R_2 = R_2/-4} \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 3/4 & -1/4 \end{array}\right]$$

A^{-1}

Conclusion:

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{bmatrix}$$

* Connection with linear maps:

A represents f

A^{-1} " g

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \end{tikzcd}$$

$AA^{-1} = A^{-1}A = I_n$

$f \circ g = g \circ f = id$ (identity map on $\mathbb{R}^n$)

g is the inverse map of f, denoted by $g = f^{-1}$.