Lecture 9 (1/28/2019)

* Application of inverse matrix:

- Represent inverse map:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "f \,\dots\, A"] & \mathbb{R}^n \arrow[l, bend left, "f^{-1} \,\dots\, A^{-1}"] \end{tikzcd}$$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x+2y, x+3y)$

Is f invertible? What is f^{-1}?

f is represented by matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Check if A is invertible:

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \quad \text{REF}$$

both cols contain pivot entries

A is invertible. Thus, f is invertible.

Continue to do row operations:

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \quad \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{A^{-1}}$$

Thus,
$$f^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 2y \\ y \end{bmatrix}$$

Conclusion, $f^{-1}(x, y) = (x - 2y, y)$

- Solve system of linear equations:

$$Ax = b$$

If A is square (# of unknowns = # eqs) and invertible

then $x = A^{-1} b$ (unique solution).

Why? Multiply both sides of $Ax = b$ by A^{-1} on the left:

$$\underbrace{A^{-1} A}_{I_n} x = A^{-1} b \implies x = A^{-1} b$$

Ex : $\begin{cases} x + 2y = 1 \\ 2x + 3y = 2 \end{cases}$

Matrix form: $$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Thus, $$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} b = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

* Questions :

- $(A+B)^{-1} \stackrel{?}{=} A^{-1} + B^{-1}$ False! Take A and B to be 1x1 matrix for example.

  $$(2+3)^{-1} \neq 2^{-1} + 3^{-1}$$

- $(cA)^{-1} \stackrel{?}{=} c^{-1} A^{-1}$ True
  (c is a nonzero number)
- $(AB)^{-1} \stackrel{?}{=} A^{-1} B^{-1}$ False!
- $(AB)^{-1} = B^{-1} A^{-1}$ true because

  $$(AB)(B^{-1} A^{-1}) = A \underbrace{B B^{-1}}_{I_n} A^{-1} = A A^{-1} = I_n$$

<u>Theorem</u>: If $A_1, A_2, \dots, A_k$ are invertible matrices of the same size then the product $A_1 A_2 \dots A_k$ is also invertible and
$$(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \dots A_2^{-1} A_1^{-1}.$$

* Inverse of 2x2 matrices :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, the matrix is <u>not</u> invertible.

Ex:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3/2 & 1/2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^{-1}$ does not exist because $1(4) - 2(2) = 0$.

The number $ad-bc$ is the means for us to check if the 2×2 matrix is invertible. It is useful information.

* Determinant: is a number associate with any square matrix. It is helpful information about the matrix. In particular,

If $\det A \neq 0$ then A is invertible.

If $\det A = 0$ then A is not invertible.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \underbrace{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}_{\text{notation}} \text{ is defined to be } ad-bc.$$

Ex:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = -2$$

$$\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 2(3) - 2(3) = 0$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

(basket weave method)

Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 2 & -1 & 3 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 0 \\ 2 & -1 \end{matrix} = 0 + 4 + (-6) - 0 - (-1) - 12 = -13$$