Final Review

Friday, March 13, 2020 10:35 AM

(7) $f(A) = AC - CA$.

$$f\Big(\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\Big) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} b & ai \\ d & ci \end{bmatrix} - \begin{bmatrix} ci & di \\ a & b \end{bmatrix}$$

$$f\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = \begin{bmatrix} b-ci & ai-di \\ d-a & ci-b \end{bmatrix}$$

$$(f(A), B) = (A, \underbrace{f^*(B)}_{??})$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be general vectors in $M_{2\times 2}(\mathbb{C})$

want to know $f^*\Big(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\Big)$.

know $f: M_{2\times 2} \to M_{2\times 2}$

$f^*: M_{2\times 2} \to M_{2\times 2}$

Write $f^*\Big(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\Big) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$.

want to find $\alpha, \beta, \gamma, \delta$ in terms of x, y, z, t.

$$LHS = (f(A), B) = \left(\begin{bmatrix} b-ci & ai-di \\ d-a & ci-b \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}\right)$$

$$LHS = (f(A), B) = \left(\begin{bmatrix} b-ci & ai-di \\ d-a & ci-b \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix} \right)$$

$$= (b-ci)\bar{x} + (ai-di)\bar{y} + (d-a)\bar{z} + (ci-b)\bar{t}.$$

$$RHS = (A, f^*(B)) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right)$$

$$= a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma} + d\bar{\delta}.$$

$$LHS = a(i\bar{y} - \bar{z}) + b(\bar{x} - \bar{t}) + c(-i\bar{x} + \bar{t}) + d(-i\bar{y} + \bar{z}).$$

$$\begin{cases} \bar{\alpha} = i\bar{y} - \bar{z} \\ \bar{\beta} = \bar{x} - \bar{t} \\ \bar{\gamma} = -i\bar{x} + \bar{t} \\ \bar{\delta} = -i\bar{y} + \bar{z} \end{cases} \xrightarrow[\text{conjugate}]{\text{take}} \begin{cases} \alpha = -iy - z \\ \beta = x - t \\ \gamma = ix + t \\ \delta = iy + z \end{cases}$$

$$\left[\begin{aligned} \bar{\alpha} &= i\bar{y} - \bar{z} \\ \bar{\bar{\alpha}} &= \overline{i\bar{y} - \bar{z}} = \overline{i\bar{y}} - \bar{\bar{z}} \\ &= -iy - z \end{aligned} \right]$$

Conclusion:

$$f^*\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix} \right) = \begin{bmatrix} -iy-z & x-t \\ ix+t & iy+z \end{bmatrix}$$

f is unitary iff $\underbrace{f^* = f^{-1}}_{\text{How to check?}}$.

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b - ci & ai - di \\ d - a & ci - b \end{bmatrix}$$

$f: M_{2\times 2} \to M_{2\times 2}$

choose the standard basis $\beta = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{e_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{e_4} \right\}$

$$[f]_\beta = \begin{bmatrix} | & & | \\ [f(e_1)]_\beta & \cdots & [f(e_4)]_\beta \\ | & & | \end{bmatrix}$$

$$f(e_1) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & i \\ -1 & 0 \end{bmatrix} = ie_2 + (-1)e_3 .$$

$$[f(e_1)]_\beta = \begin{bmatrix} 0 \\ i \\ -1 \\ 0 \end{bmatrix}$$

$$f(e_2) = f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = e_1 - e_3$$

$$[f(e_2)]_\beta = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$- - -$

$[f]_\beta = \ldots\ldots$

$[f^*]_\beta = \begin{bmatrix} 0 & & & \\ 1 & & & \\ i & & & \\ 0 & & & \end{bmatrix}$

check if product $= I_4$

(4) $G: P_2 \to P_2$

$G(u)(x) = (x+1)u'(x) - u(x).$

Let's translate the prob. into Coordinate

→ need matrix rep. of G

Choose $\mathcal{B} = \{\underbrace{x^2}_{u_1}, \underbrace{x}_{u_2}, \underbrace{1}_{u_3}\}$

$$[G]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [G(u_1)]_{\mathcal{B}} & [G(u_2)]_{\mathcal{B}} & [G(u_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$$

$$G(u_1) = (x+1)u_1'(x) - u_1(x)$$

$$= (x+1)2x - x^2 = x^2 + 2x. \to \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$G(u_2) = (x+1)\underbrace{u_2'}_{1} - \underbrace{u_2}_{x} = 1 \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G(u_3) = (x+1)\underbrace{u_3'}_{=0} - \underbrace{u_3}_{1} = -1 \to \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$[G]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

G is diagonalizable iff $[G]_{\mathcal{B}}$ is diagonalizable.

$[G]_{\mathcal{B}}$ is lower triangular.

⇝ $\lambda = 1, 0, -1$ are eigenvalues.

⇝ $[G]_{\mathcal{B}}$ is diagonalizable.

$$[G]_{\mathcal{B}} = P \underset{\text{diag..}}{D} P^{-1} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Q

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

v_1 ... eigenv. of λ_1

v_2 ...... λ_2

v_3 ...... λ_3

Find v_1: $(Q - \underbrace{\lambda_1}_{1} I_3) v = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\quad\right]$$

solve ... get v_1.

Find v_2: $(Q - \underbrace{\lambda_2}_{0} I_3) v = 0$

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Basis of $\mathbb{R}^3$ that diagonalizes Q is $\{v_1, v_2, v_3\}$

$G : P_2 \to P_2$ ---------- $Q : \mathbb{R}^3 \to \mathbb{R}^3$.

Ex: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\mathcal{C} = \{v_1, v_2, v_3\}$

$\mathcal{B}$ that diagonalizes G is $\{w_1, w_2, w_3\}$

$w_1 = x^2 + 2x + 3$ $\qquad [w_1]_{\mathcal{B}} = v_1$

$w_2 = 2x^2 + 4x + 5$ $\qquad [w_2]_{\mathcal{B}} = v_2$

$w_3 = 0x^2 + 1x + 1.$ $\qquad [w_3]_{\mathcal{B}} = v_3.$