# Homework 1

**Answer Key**

1. Let $V$ be a vector space over a field $F$ (which is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$). Use the axioms of vector space to show the following properties. Make sure to mention which axiom(s) you use.

   (a) (Cancellation law) If $u_1, u_2, v \in V$ and $u_1 + v = u_2 + v$, then $u_1 = u_2$.

   **Solution:** Let $u_1, u_2, v \in V$ and assume

   $$u_1 + v = u_2 + v.$$

   Let $w$ be an additive inverse of $v$ ($w$ exists by the additive inverse axiom). By adding $w$ to both sides we get

   $$(u_1 + v) + w = (u_2 + v) + w$$

   Notice that we need to keep $(u_1 + v)$ and $(u_2 + v)$ grouped, since addition is technically only defined between two vectors and not three. Now apply the associativity of addition axiom:

   $$u_1 + (v + w) = u_2 + (v + w)$$

   By the definition of the additive inverse, $v + w = 0$ where $0$ is an additive identity or "neutral element". We can then write

   $$u_1 + 0 = u_2 + 0.$$

   We use the additive identity axiom to simplify

   $$u_1 + 0 = u_1 \qquad \text{and} \qquad u_2 + 0 = u_2.$$

   If your definition of the additive identity axiom uses the opposite ordering (i.e., $0 + v = v$ instead of $v + 0 = v$) then you technically need to apply commutativity of addition before making the above reductions. Now apply these reductions to both sides of the equation to conclude

   $$u_1 = u_2.$$

   (b) (Uniqueness of zero element) If $a$ and $b$ are neutral elements of $V$, i.e.

   $$a + v = v \quad \forall v \in V,$$
   $$b + v = v \quad \forall v \in V,$$

   then $a = b$.

   **Solution:** Let $a$ and $b$ be neutral elements or "additive identities" of $V$ as defined above and consider $v = b$. Then

   $$a + b = b, \text{ and}$$
   $$b + b = b.$$

   Therefore we can write

   $$a + b = b + b.$$

   By applying the cancellation law (problem 1.(a)) we can cancel $b$ to conclude that

   $$a = b$$

(c) (Scaling by 0)

$$0v = 0 \quad \forall v \in V.$$

**Solution:** Note that the zero on the left represents a scalar, and the zero on the right represents a vector. To make this proof easier to follow, I will denote the scalar zero by 0 and the vector zero by $\vec{0}$.

Let $v \in V$ and consider the fact that $(0 + 0) = 0$ (remember, this is an equation of scalars). Now we can write

$$(0 + 0)v = 0v$$

Since $0v = \vec{0} + 0v$ by the additive identity axiom, we can rewrite the left-hand side of the equation:

$$(0 + 0)v = \vec{0} + 0v$$

By using distribution over scalar addition we can rewrite the right-hand side:

$$0v + 0v = \vec{0} + 0v$$

Now we can use the cancellation law (problem 1.(a)) to conclude

$$0v = 0.$$

(d) (Additive inverse) If $v, w \in V$ satisfy $v + w = 0$ then $w = (-1)v$ (vector $v$ scaled by factor $-1$).

**Solution:** Let $v, w \in V$ and suppose $v + w = 0$. By problem 1.(c) we can write $0 = 0v$, so

$$v + w = 0v$$

Now use the scalar equation $0 = (1 + [-1])$ to write

$$v + w = (1 + [-1])v.$$

Apply distribution over scalar addition:

$$v + w = 1v + (-1)v.$$

The multiplicative identity axiom gives $1v = v$, so we can rewrite the equation as

$$v + w = v + (-1)v.$$

Now you might want to use problem 1.(a), but we have to be careful here. Notice that when we proved the cancellation law, the vector we were canceling was on the right side of the addition, but in this case we want to cancel the vector on the left side of the addition. Use commutativity of addition to switch the order of both sides of the equation first:

$$w + v = (-1)v + v.$$

We can now use problem 1.(a) to conclude

$$w = (-1)v.$$

2. On the set of complex numbers $\mathbb{C}$, we define another product rule as follows:

$$z * v = \bar{z}v \in \mathbb{C}.$$

The star denotes the new product rule. The product on the right hand side is the usual product of complex numbers. Here $\bar{z}$ denotes the complex conjugate of z. Show that $V = \mathbb{C}$ is a vector space over $F = \mathbb{C}$ under the usual addition and the new product rule.

**Solution:** To show that this is a vector space we need to show that it satisfies the vector space axioms. Since we are still using ordinary addition, all of the addition axioms still hold just as they did under the standard vector operations in $\mathbb{C}$. We only need to check the scaling and distribution axioms.

Recall that every element $z \in \mathbb{C}$ can be written at $z = a + bi$ for $a, b \in \mathbb{R}$. The complex conjugate is then defined as

$$\bar{z} = a - bi.$$

We now prove the axioms:

(i) **Closed under scalar multiplication:** Let $z \in F = \mathbb{C}$ and $v \in V = \mathbb{C}$. Then $\bar{z}$ and $v$ are both complex numbers, so their product $\bar{z}v = z * v$ is also a complex number.

(ii) **Multiplicative identity:** Consider $z = a + bi \in \mathbb{C}$. If $z = 1$ then $a = 1$ and $b = 0$, so

$$\bar{1} = 1 - 0i = 1.$$

Therefore

$$\begin{aligned} 1 * z &= \bar{1}z \\ &= 1z \\ &= z. \end{aligned}$$

(iii) **Associativity of scalar multiplication:** Let $w, z \in F = \mathbb{C}$. We will need the fact that

$$\overline{wz} = \bar{w}\bar{z}.$$

To prove this, let $w = a + bi$ and $z = c + di$. First expand $\overline{wz}$:

$$\begin{aligned} \overline{wx} &= \overline{(ac - bd) + (bc + ad)i} \\ &= (ac - bd) - (bc + ad)i \end{aligned}$$

Now expand $\bar{w}\bar{z}$:

$$\begin{aligned} \bar{w}\bar{z} &= (a - bi)(c - di) \\ &= ac - adi - bci - bd \\ &= (ac - bd) - (bc + ad)i. \end{aligned}$$

From this we conclude that $\overline{wz} = \bar{w}\bar{z}$.

We can now prove associativity of scalar multiplication. Let $w, z \in F = \mathbb{C}$ and $v \in V = \mathbb{C}$. Then

$$\begin{aligned} (wz) * v &= \overline{wz}v \\ &= \bar{w}\bar{z}v \\ &= \bar{w}(z * v) \\ &= w * (z * v). \end{aligned}$$

(iv) **Distribution over vector addition:** Let $z \in F = \mathbb{C}$ and $u, v \in V = \mathbb{C}$. Then

$$\begin{aligned} z * (u + v) &= \bar{z}(u + v) \\ &= \bar{z}u + \bar{z}v \\ &= z * u + z * v. \end{aligned}$$

(v) **Distribution over scalar addition:** Let $w, z \in F = \mathbb{C}$. We will need the fact that

$$\overline{w + z} = \bar{w} + \bar{z}.$$

To prove this, let $w = a + bi$ and $z = c + di$. Then

$$\begin{aligned} \overline{w + z} &= \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i \\ &= (a - bi) + (c - di) \\ &= \bar{w} + \bar{z}. \end{aligned}$$

We can now prove distribution over scalar addition. Let $w, z \in F = \mathbb{C}$ and $v \in V = \mathbb{C}$. Then

$$\begin{aligned} (w + z) * v &= (\overline{w + z})v \\ &= (\bar{w} + \bar{z})v \\ &= \bar{w}v + \bar{z}v \\ &= w * v + z * v. \end{aligned}$$

3. Let $F$ be a field of numbers. Put

$$V = \{A \in M_{2\times 2}(F) : A + A^T = 0\}.$$

(a) Show that $V$ is a vector space over $F$. Here $A^T$ denotes the transpose of matrix $A$.

**Solution:** Notice that $V$ is a subset of the vector space $M_{2\times 2}(F)$, so we only need to show that $V$ is a **subspace** of $M_{2\times 2}(F)$. This requires proving three axioms. Before we begin, we note two facts from MTH 341 that will be useful:

- $(A + B)^T = A^T + B^T$ for all $A, B \in M_{2\times 2}(F)$.
- $(\lambda A)^T = \lambda(A^T)$ for all $A \in M_{2\times 2}(F)$ and $\lambda \in F$.

(i) **V contains the zero vector:** In $M_{2\times 2}(F)$, the zero vector is the $2 \times 2$ zero matrix. It is easy to check that this is in $V$:

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

(ii) **V is closed under addition:** Let $A, B \in V$. This means that $A + A^T = 0$ and $B + B^T = 0$ where 0 represents the zero matrix. Now

$$\begin{aligned} (A + B) + (A + B)^T &= A + B + A^T + B^T \\ &= (A + A^T) + (B + B^T) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

so $(A + B) \in V$.

(iii) **V is closed under scalar multiplication:** Let $A \in V$ and $\lambda \in F$. Then

$$\begin{aligned} (\lambda A) + (\lambda A)^T &= \lambda(A) + \lambda(A^T) \\ &= \lambda(A + A^T) \\ &= \lambda 0 \\ &= 0, \end{aligned}$$

so $(\lambda A) \in V$.

Since $V$ is a subspace of $M_{2\times 2}(F)$, $V$ is a vector space.

(b) Find a basis and the dimension of $V$.

**Solution:** It is helpful to rewrite the elements of $V$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in F.$$

Then

$$\begin{aligned} A + A^T &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} \end{aligned}$$

If $A \in V$ then

$$\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives the equations

$$\begin{aligned} 2a &= 0, \\ b + c &= 0, \\ 2d &= 0. \end{aligned}$$

From this we can deduce that $a = d = 0$ and $c = -b$, so $V$ can be rewritten as

$$\begin{aligned} V &= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in F \right\} \\ &= \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : b \in F \right\} \end{aligned}$$

Our proposed basis for $V$ is

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

(note that $\mathcal{B}$ is a set, not a matrix). We need to check the properties of a basis:

(i) It is easy to check that $\mathcal{B} \subset V$.

(ii) Our rewriting of $V$ shows that $\operatorname{span}(\mathcal{B}) = V$.

(iii) Since $\mathcal{B}$ has exactly one nonzero element, $\mathcal{B}$ is linearly independent.

Therefore $\mathcal{B}$ is a basis for $V$. Since $\mathcal{B}$ has exactly one element, the dimension of $V$ is

$$\dim(V) = 1.$$

4. Let $V = \mathbb{Q}^{(1,3)\cap\mathbb{Q}}$, which is the set of all functions from $(1,3)\cap\mathbb{Q}$ to $\mathbb{Q}$. Recall that $V$ is a vector space over $F = \mathbb{Q}$.

   (a) Do the functions $f(x) = \frac{x}{x-2}$ and $g(x) = \sqrt{x}$ belong to $V$?
   **Solution:** These functions do not belong to $V$.

   - The function $f(x)$ is not defined at $x = 2 \in (1,3)\cap\mathbb{Q}$, so it cannot be a function from $(1,3)\cap\mathbb{Q}$.
   - It is a well known fact that $\sqrt{2} \notin \mathbb{Q}$ (I won't give the proof here, but you can look it up online). Therefore $g(2) \notin \mathbb{Q}$, so $g(x)$ cannot be a function from $(1,3)\cap\mathbb{Q}$ to $\mathbb{Q}$.

   (b) Consider three functions $f_1(x) = x-1$, $f_2(x) = x$, and $f_3(x) = 1/x$. They are vectors in $V$. Show that $f_1, f_2, f_3$ are linearly independent.
   **Solution:** To prove this set is linearly independent, set a linear combination equal to 0:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

Note that this equation must hold for all $x \in (1,3)\cap\mathbb{Q}$. We want to show that $c_1 = c_2 = c_3 = 0$. To do this, we can plug in three different values for $x$ to get three equations with three unknowns. We must choose values in $(1,3)\cap\mathbb{Q}$. I chose

$$x = \frac{3}{2}, \qquad x = 2, \qquad x = \frac{5}{2}.$$

Plugging each of these into our equation gives the following system of equations:

$$\begin{aligned} \tfrac{1}{2}c_1 + \tfrac{3}{2}c_2 + \tfrac{2}{3}c_3 &= 0 \\ c_1 + 2c_2 + \tfrac{1}{2}c_3 &= 0 \\ \tfrac{3}{2}c_1 + \tfrac{5}{2}c_2 + \tfrac{2}{5}c_3 &= 0 \end{aligned}$$

It is probably easier to work with whole numbers rather than fractions, so multiply the first equation by 6, the second equation by 2, and the third equation by 10:

$$\begin{aligned} 3c_1 + 9c_2 + 4c_3 &= 0 \\ 2c_1 + 4c_2 + c_3 &= 0 \\ 15c_1 + 25c_2 + 4c_3 &= 0 \end{aligned}$$

We can write this system in augmented matrix form:

$$\left[\begin{array}{ccc|c} 3 & 9 & 4 & 0 \\ 2 & 4 & 1 & 0 \\ 15 & 25 & 4 & 0 \end{array}\right]$$

The next step is to reduce this matrix to find that $c_1 = c_2 = c_3 = 0$. The row reduction is tedious, so I will use MATLAB:

```
A = [3   9  4  0;
     2   4  1  0;
    15  25  4  0];
R = rref(A);
```

You should get

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

showing that $c_1 = c_2 = c_3 = 0$. Therefore the vectors $f_1, f_2, f_3$ are linearly independent.