Homework 1

Due 01/17/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Let V be a vector space over a field F (which is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$). Use the axioms of vector space to show the following properties. Make sure to mention which axiom(s) you use.

   (a) (Cancellation law) If $u_1,\ u_2,\ v \in V$ and $u_1 + v = u_2 + v$, then $u_1 = u_2$.

   (b) (Uniqueness of zero element) If a and b are neutral elements of V, i.e.

   $$a + v = v \quad \forall v \in V,$$
   $$b + v = v \quad \forall v \in V,$$

   then $a = b$.
   Note: Because the neutral element is unique, it is denoted by 0 and is called the zero vector.

   (c) (Scaling by 0)
   $$0v = 0 \quad \forall v \in V.$$

   (d) (Additive inverse) If $v,\ w \in V$ satisfy $v + w = 0$ then $w = (-1)v$ (vector v scaled by factor -1).
   Note: the additive inverse of v is denoted as $-v$.

2. On the set of complex numbers $\mathbb{C}$, we define another product rule as follows:

   $$z * v = \bar{z}v \qquad \forall z, v \in \mathbb{C}.$$

   The star denotes the new product rule. The product on the right hand side is the usual product of complex numbers. Here $\bar{z}$ denotes the complex conjugate of z. Show that $V = \mathbb{C}$ is a vector space over $F = \mathbb{C}$ under the usual addition and the new product rule.

3. Let F be a field of numbers. Put

   $$V = \{A \in M_{2\times 2}(F) : \ A + A^T = 0\}.$$

   (a) Show that V is a vector space over F. Here A^T denotes the transpose of matrix A.

   (b) Find a basis and the dimension of V.

Do the following problem for 6 bonus points.

4. Let $V = \mathbb{Q}^{(1,3)\cap\mathbb{Q}}$, which is the set of all functions from $(1, 3) \cap \mathbb{Q}$ to $\mathbb{Q}$. Recall that V is a vector space over $F = \mathbb{Q}$.

   (a) Do the functions $f(x) = \frac{x}{x-2}$ and $g(x) = \sqrt{x}$ belong to V ?

   (b) Consider three functions $f_1(x) = x - 1$, $f_2(x) = x$, and $f_3(x) = 1/x$. They are vectors in V. Show that $f_1,\ f_2,\ f_3$ are linearly independent.