Homework 2
Answer Key

1. Let U, V be subspaces of a vector space W.

   (a) The subspace sum of U and V is $U + V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U; \mathbf{v} \in V\}$. Show that $U + V$ is a subspace of W.

   Solution: We must prove the three subspace axioms:

   - ***V* contains 0:** Since U and V are subspaces, they both contain 0, so
   $$0 = 0 + 0 \in U + V.$$
   - ***V* is closed under addition:** Let $\mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{u}_2 + \mathbf{v}_2$ be arbitrary elements of $U + V$ where $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then
   $$(\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \in U + V.$$
   - ***V* is closed under scalar multiplication:** Let $\mathbf{u} + \mathbf{v}$ be an arbitrary element of $U + V$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$, and let λ be an arbitrary scalar. Then
   $$\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v} \in U + V$$

   (b) Let $\mathbf{x} \in W$, but $\mathbf{x} \notin V$. Show that for all $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{x} \notin V$.

   Solution: We will use proof by contradiction. Suppose that there exists $\mathbf{v} \in V$ such that $\mathbf{v} + \mathbf{x} \in V$. Then
   $$\mathbf{x} = -\mathbf{v} + (\mathbf{v} + \mathbf{x}) \in V,$$
   but this contradicts our assumption that $\mathbf{x} \notin V$.

   (c) Show that the union $U \cup V$ is a subspace if and only if $U \subseteq V$ or $V \subseteq U$.

   Solution: This is an if and only if statement, so we must prove the implication in two directions.

   (i) ($\Leftarrow$) Assume $U \subseteq V$. Then $U \cup V = V$ which was assumed to be a subspace of W. Similarly, if $V \subseteq U$, then $U \cup V = U$ is a subspace of W by assumption.

   (ii) ($\Rightarrow$) Assume $U \cup V$ is a subspace. For this direction we will use a proof by contradiction. Suppose that $U \not\subseteq V$ and $V \not\subseteq U$. Then there exists $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $\mathbf{u} \notin V$ and $\mathbf{v} \notin U$. By problem 1(b) we have that
   $$\mathbf{u} + \mathbf{v} \notin U \qquad \text{and} \qquad \mathbf{u} + \mathbf{v} \notin V,$$
   so $\mathbf{u} + \mathbf{v} \notin U \cup V$. This contradicts the assumption that $U \cup V$ is a subspace (since subspaces are closed under addition).

(d) The subspace sum $U+V$ is a direct sum (denoted $U \oplus V$) if each element in $U+V$ can be written in only one way as a sum $\mathbf{u}+\mathbf{v}$ where $\mathbf{u} \in U,\ \mathbf{v} \in V$. Show that $U+V$ is a direct sum if and only if $U \cap V = \{0\}$.

Solution: Again, this is an if and only if statement, so we must prove the implication in two directions.

(i) ($\Leftarrow$) Assume $U \cap V = \{0\}$ and let $\mathbf{u}_1+\mathbf{v}_1$ and $\mathbf{u}_2+\mathbf{v}_2$ be arbitrary elements of $U+V$ where $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$. We want to show that if $\mathbf{u}_1+\mathbf{v}_1$ and $\mathbf{u}_2+\mathbf{v}_2$ represent the same element of $U+W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$.
Suppose that

$$\mathbf{u}_1 + \mathbf{v}_1 = \mathbf{u}_2 + \mathbf{v}_2.$$

Rearranging this equation gives

$$\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_1.$$

Since $\mathbf{u}_1 - \mathbf{u}_2 \in U$ and $\mathbf{v}_2 - \mathbf{v}_1 \in V$, both sides of this equation must be in $U \cap V = \{0\}$. Then

$$\mathbf{u}_1 - \mathbf{u}_2 = 0,$$
$$\mathbf{v}_2 - \mathbf{v}_1 = 0,$$

so $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$ as desired.

(ii) ($\Rightarrow$) Assume each element of $U+V$ can be written in only one way as a sum $\mathbf{u}+\mathbf{v}$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$. We will use proof by contradiction for this direction. Suppose that $U \cap V \neq \{0\}$, so there exists a nonzero $\mathbf{w} \in U \cap V$. Then there is more than one way to write $0 \in U+V$ as $\mathbf{u}+\mathbf{v}$. For example, we could let

$$\mathbf{u} = \mathbf{v} = 0,$$

or we could let

$$\mathbf{u} = \mathbf{w} \text{ and } \mathbf{v} = -\mathbf{w}.$$

This contradicts our assumption that each element of $U+V$ can be written in only one way as a sum $\mathbf{u}+\mathbf{v}$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

2. Show that $1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3$ forms a basis of $\mathbb{P}_3(\mathbb{F})$ (polynomials over the field $\mathbb{F}$ of degree at most 3).

Solution: Let

$$B = \{1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3\}.$$

To show that B is a basis for $\mathbb{P}_3(\mathbb{F})$ we need to show that it spans $\mathbb{P}_3(\mathbb{F})$ and is linearly independent.

Notice that

$$\begin{aligned} 1 &= 1, \\ x &= (1+x) - (1), \\ x^2 &= (1+x+x^2) - (1+x), \\ x^3 &= (1+x+x^2+x^3) - (1+x+x^2), \end{aligned}$$

so $1, x, x^2, x^3 \in \operatorname{span}(B)$. Since $\{1, x, x^2, x^3\}$ is a basis for $\mathbb{P}_3(\mathbb{F})$ it spans all of $\mathbb{P}_3(\mathbb{F})$, so we must also have that B also spans all of $\mathbb{P}_3(\mathbb{F})$.

To see that B is linearly independent, notice that B is a set of four vectors that spans the 4-dimensional space $\mathbb{P}_3(\mathbb{F})$. Therefore B is a basis for $\mathbb{P}_3(\mathbb{F})$, so B is linearly independent.

The fact that a set of n vectors spanning an n-dimensional space is a basis was covered on recitation worksheet 3 and is also covered in the book *Linear Algebra Done Right*.

3. Let U, V be vector spaces and $T: U \to V$ be a linear transformation.

(a) Show that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in U$ and if $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_k$ are linearly independent then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Solution: Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in U$ and that $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_k$ are linearly independent. Set a linear combination of the $\mathbf{v}_i$ equal to zero:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0 \qquad \text{for scalars } c_1, c_2, \ldots, c_k.$$

We want to show that $c_i = 0$ for all $i = 1, \ldots, k$. Apply the transformation T to both sides of the equation and use linearity:

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) &= T(0) \\ &\downarrow \\ c_1T\mathbf{v}_1 + c_2T\mathbf{v}_2 + \cdots + c_kT\mathbf{v}_k &= 0 \end{aligned}$$

Since $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_k$ are linearly independent, we must have that $c_i = 0$ for all $i = 1, \ldots, k$.

(b) T is <u>injective</u> if for every $\mathbf{u}, \mathbf{v} \in U$, $T\mathbf{u} = T\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$. An injective linear transformation is called a monomorphism. Show that T is a monomorphism if and only if the null space (aka kernel) of T is $\{0\}$.

Solution: we must prove the implication in both directions.

(i) ($\Rightarrow$) Assume T is a monomorphism. Let $\mathbf{u} \in \text{null}(T)$. Then by the definition of the null space, $T(\mathbf{u}) = 0$. Therefore

$$T\mathbf{u} = 0 = T(0).$$

Since T is a monomorphism (i.e., is injective) we must have $\mathbf{u} = 0$, so $\text{null}(T) = \{0\}$.

(ii) ($\Leftarrow$) Assume $\text{null}(T) = \{0\}$. Let $\mathbf{u}, \mathbf{v} \in U$ such that $T\mathbf{u} = T\mathbf{v}$. Then

$$\begin{aligned} T(\mathbf{u} - \mathbf{v}) &= T\mathbf{u} - T\mathbf{v} \\ &= 0, \end{aligned}$$

so $\mathbf{u} - \mathbf{v} \in \text{null}(T)$. Since $\text{null}(T) = \{0\}$, we have $\mathbf{u} - \mathbf{v} = 0$, so $\mathbf{u} = \mathbf{v}$ as desired.

(c) Show that if T is a monomorphism and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in U$ are linearly independent then $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_k$ are linearly independent.

Solution: Assume T is a monomorphism and suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in U$ are linearly independent. Set a linear combination of the $T\mathbf{v}_i$ equal to zero:

$$c_1T\mathbf{v}_1 + c_2T\mathbf{v}_2 + \cdots + c_kT\mathbf{v}_k = 0 \qquad \text{for scalars } c_1, c_2, \ldots, c_k.$$

We want to show that $c_i = 0$ for all $i = 1, \ldots, k$. Using linearity, we can rewrite the above equation as

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = T(0).$$

Since T is injective, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0.$$

Finally, since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent we have $c_i = 0$ for all $i = 1, \ldots, k$.

4. Let V be a vector space over a field $\mathbb{F}$ and $T \in \mathcal{L}(V, \mathbb{F})$. Suppose $\mathbf{v} \in V$ is not in $\ker(T)$. Show that $V = \ker(T) \oplus \{c\mathbf{v} \mid c \in \mathbb{F}\}$.

Solution: Since $\ker(T)$ and $\{c\mathbf{v} \mid c \in \mathbb{F}\}$ are subsets of V, their sum is also a subset of V. We now want to show that $V \subseteq \ker(T) + \{c\mathbf{v} \mid c \in \mathbb{F}\}$.

Let $\mathbf{u}$ be an arbitrary element of V, and consider the vector

$$\mathbf{w} = \mathbf{u} - \frac{T\mathbf{u}}{T\mathbf{v}}\mathbf{v} \in V$$

(note that $T\mathbf{v} \neq 0$ since $\mathbf{v} \notin \ker(T)$). If we apply T to $\mathbf{w}$ we get

$$\begin{aligned} T\mathbf{w} &= T\left(\mathbf{u} - \frac{T\mathbf{u}}{T\mathbf{v}}\mathbf{v}\right) \\ &= T\mathbf{u} - \frac{T\mathbf{u}}{T\mathbf{v}}T\mathbf{v} \\ &= T\mathbf{u} - T\mathbf{u} \\ &= 0, \end{aligned}$$

so $\mathbf{w} \in \ker(T)$. Therefore

$$\mathbf{u} = \mathbf{w} - \frac{T\mathbf{u}}{T\mathbf{v}}\mathbf{v} \in \ker(T) + \{c\mathbf{v} \mid c \in \mathbb{F}\},$$

so $V \subseteq \ker(T) + \{c\mathbf{v} \mid c \in \mathbb{F}\}$ and hence $V = \ker(T) + \{c\mathbf{v} \mid c \in \mathbb{F}\}$.

We now want to show that this sum is direct. Let $\mathbf{u} \in \ker(T) \cap \{c\mathbf{v} \mid c \in \mathbb{F}\}$. Since $\mathbf{u} \in \{c\mathbf{v} \mid c \in \mathbb{F}\}$ we can write

$$\mathbf{u} = \lambda\mathbf{v} \qquad \text{for some } \lambda \in \mathbb{F}.$$

Since $\mathbf{u} \in \ker(T)$ we have

$$\begin{aligned} 0 &= T\mathbf{u} \\ &= T(\lambda\mathbf{v}) \\ &= \lambda T\mathbf{v}. \end{aligned}$$

$\mathbf{v}$ was assumed to not be in $\ker(T)$, so $T(\mathbf{v}) \neq 0$. Thus we can divide both sides by $T\mathbf{v}$ to get $\lambda = 0$. Therefore

$$\mathbf{u} = 0\mathbf{v} = 0,$$

so $\ker(T) \cap \{c\mathbf{v} \mid c \in \mathbb{F}\} = \{0\}$. By problem 1(d) the subspace sum is direct, and we can write

$$V = \ker(T) \oplus \{c\mathbf{v} \mid c \in \mathbb{F}\}.$$