Homework 2

Due 01/27/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Let U and V be subspaces of a vector space W.

   (a) The *sum* of U and V, denoted by $U + V$, is defined as the set $U + V = \{u + v : \ u \in U,\ v \in V\}$. Show that $U + V$ is a subspace of W.

   (b) Let w be a vector in W but not in V. Show that

   $$w + v \notin V \quad \forall v \in V.$$

   Hint: proof by contradiction. (Suppose that the conclusion is false. Then use valid arguments to find a contradiction.)

   (c) Show that the union $U \cup V$ is a subspace of W if and only if either $U \subset V$ or $V \subset U$. Hint: proof by contradiction.

   (d) The sum $U + V$ is said to be a *direct sum* if each vector in $U + V$ can be written in *only one way* as $u + v$ where $u \in U$ and $v \in V$. ("Only one way" means that if $x \in U + V$ and $x = u_1 + v_1 = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$ then $u_1 = u_2$ and $v_1 = v_2$.) Show that $U + V$ is a direct sum if and only if $U \cap V = \{0\}$.

2. Let F be a field of numbers ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). Show that the polynomials 1, $1 + x$, $1 + x + x^2$, $1 + x + x^2 + x^3$ form a basis of $P_3(F)$ (the space of all polynomials of degree ≤ 3 with coefficients in F).

3. Let U and V be vector spaces over a field F. Let $f : U \to V$ be a linear map.

   (a) Let u_1, u_2,$\ldots u_k \in U$. Show that if $f(u_1)$, $f(u_2)$,$\ldots$,$f(u_k)$ are linearly independent then $u_1, u_2, \ldots, u_k$ are also linearly independent.

   (b) f is said to be *injective*, or *monomorphic*, if $f(u) = f(v)$ implies $u = v$. Show that f is monomorphic if and only if $\text{null}(f) = \{0\}$.

   (c) Suppose f is monomorphic. Show that f preserves linear independence. That is to show: if $u_1, u_2, \ldots, u_k \in U$ are linearly independent then $f(u_1), f(u_2), \ldots, f(u_k)$ are also linearly independent.

Do the following problem for 6 bonus points.

4. Let V be a vector space over F and $f : V \to F$ be a linear map. Let v be a vector in V but not in $\text{null}(f)$. Show that $V = \text{null}(f) + \text{span}\{v\}$. Is this sum a direct sum?