Homework 3
Answer Key

1. Let $G\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $G(f)(x) = (x+1)f'(x) - 2f(x)$.

   (a) Show that G is a linear map.
   Solution: We must show the two properties of a linear map:

   - **Additivity:** Let $u, v \in P_2(\mathbb{R})$. Then

   $$\begin{aligned} G(u+v)(x) &= (x+1)(u(x)+v(x))' - 2(u(x)+v(x)) \\ &= (x+1)(u'(x)+v'(x)) - 2u(x) - 2v(x)) \\ &= \big[(x+1)u'(x) - 2u(x)\big] + \big[(x+1)v'(x) - 2v(x)\big] \\ &= G(u)(x) + G(v)(x). \end{aligned}$$

   - **Homogeneity:** Let $f \in P_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then

   $$\begin{aligned} G(\lambda f)(x) &= (x+1)(\lambda f(x))' - 2(\lambda f(x)) \\ &= (x+1)\lambda f'(x) - 2\lambda f(x) \\ &= \lambda\big[(x+1)f'(x) - 2f(x)\big] \\ &= \lambda G(f)(x). \end{aligned}$$

   (b) Find a basis for $\ker(G)$ (a.k.a $\text{null}(G)$). What is the dimension of $\ker(G)$?
   Solution: Let $f \in \ker(G)$. Since $f \in P_2(\mathbb{R})$ we can write

   $$f(x) = ax^2 + bx + c \qquad \text{for some } a, b, c \in \mathbb{R}.$$

   Now since f is in $\ker(G)$ we have $G(f)(x) = 0$, so

   $$\begin{aligned} 0 &= G(f)(x) \\ &= G(ax^2 + bx + c) \\ &= (x+1)(2ax+b) - 2(ax^2+bx+c) \\ &= (2a-b)x + (b-2c). \end{aligned}$$

   Since x and 1 are linearly independent, the only way to get $0 = (2a-b)x + (b-2c)1$ is if

   $$2a - b = 0 \qquad \text{and} \qquad b - 2c = 0.$$

   Therefore $2a = b = 2c$, so $f(x)$ can be written as

   $$\begin{aligned} f(x) &= ax^2 + 2ax + a \\ &= a(x^2 + 2x + 1). \end{aligned}$$

   Since $f(x)$ was an arbitrary element of $\ker(G)$, we can write every element of $\ker(G)$ as $a(x^2+2x+1)$ for some $a \in \mathbb{R}$. That is,

   $$\begin{aligned} \ker(G) &= \{a(x^2+2x+1) : a \in \mathbb{R}\} \\ &= \text{span}(\{x^2+2x+1\}) \end{aligned}$$

   Now $\{x^2+2x+1\}$ has only one (nonzero) element, so it is linearly independent. It is therefore a basis for $\ker(G)$. The dimension of $\ker(G)$ is

   $$\text{nullity}(G) = \dim(\ker(G)) = 1.$$

(c) Find a basis of $G(P_2(\mathbb{R}))$ (the range of G). What is the rank of G?

Solution: The range of G is the set of all outputs of the map G. That is

$$\text{range}(G) = \{G(f(x)) : f(x) \in P_2(\mathbb{R})\}.$$

If $f(x) \in P_2(\mathbb{R})$ we can write

$$f(x) = ax^2 + bx + c, \qquad \text{for some } a, b, c \in \mathbb{R},$$

so

$$\begin{aligned}\text{range}(G) &= \{G(ax^2 + bx + c) : a, b, c \in \mathbb{R}\}\\ &= \{(2a - b)x + (b - 2c) : a, b, c \in \mathbb{R}\}.\end{aligned}$$

Since $(2a - b)x + (b - 2c)$ is a first-degree polynomial, it must be contained in the span of $\{x, 1\}$. That is,

$$\text{range}(G) \subseteq \text{span}(\{x, 1\})$$

Now notice that letting $a = b = c = 1$ gives

$$(2a - b)x + (b - 2c) = x,$$

so $x \in \text{range}(G)$. Similarly, letting $a = 1$, $b = 2$, and $c = \frac{1}{2}$ gives

$$(2a - b)x + (b - 2c) = 1,$$

so $1 \in \text{range}(G)$. Therefore $\text{span}(\{x, 1\}) \subseteq \text{range}(G)$, so

$$\text{range}(G) = \text{span}(\{x, 1\}).$$

Since $\{x, 1\}$ is linearly independent, it is a basis for $\text{range}(G)$. The rank of G is

$$\text{rank}(G) = \dim(\text{range}(G)) = 2.$$

(d) Is G a monomorphism, epimorphism (an onto linear map), and/or an isomorphism?

Solution:

- We showed in homework 2 that G is a monomorphism if and only if $\ker(G) = \{0\}$. Since this is not true by part (b) above, G is not a monomorphism.
- To be an epimorphism we would need the range of G to be equal to the codomain. That is, we would need $\text{range}(G) = P_2(\mathbb{R})$. Since this is not true by part (c) above, G is not an epimorphism.
- To be an isomorphism, G needs to be both a monomorphism and an epimorphism. Since it is neither a monomorphism nor an epimorphism, G is not an isomorphism.

2. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : a+b+c+id = 0 \right\}$ and let $H\colon V \to P_2(\mathbb{C})$ be given by

$$H\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)z^2 + (b+c)z + (c+d), \text{ which is linear.}$$

(a) Show that V is a subspace of $M_{2\times 2}(\mathbb{C})$.

Solution: A valid way to solve this problem is to prove the three subspace axioms:

- V is closed under addition,
- V is closed under scalar multiplication,
- V contains the zero vector.

However, I am going to use a different argument. I will show that V is the span of a set of vectors. The span of any set of vectors is always a vector space.

Let A be an arbitrary element of V, so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{C}$$

with

$$a + b + c + id = 0.$$

From this condition we can solve for a:

$$a = -b - c - id,$$

so we can write A as

$$A = \begin{bmatrix} -b-c-id & b \\ c & d \end{bmatrix}, \qquad b, c, d \in \mathbb{C}$$

Since A was an arbitrary element of V, we can rewrite V as

$$\left\{ \begin{bmatrix} -b-c-id & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{C} \right\} = \left\{ b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} : b, c, d \in \mathbb{C} \right\}$$

Notice that

$$b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

is a linear combination of three matrices, so V is the set of all linear combinations of these three matrices. That is,

$$V = \operatorname{span}\left(\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\} \right),$$

so V is a vector space.

(b) Find a basis of V.

Solution: From our work in part (a) we found that $V = \text{span}(B)$ where

$$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

We now want to show that B is linearly independent. Set a linear combination of the vectors equal to 0:

$$b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} -b - c - id & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives $b = 0$, $c = 0$, and $d = 0$. Therefore B is linearly independent, so it is a basis for V.

(c) Find a matrix representation of H. Clearly identify the bases that you are using.

Solution: For V we will use the basis

$$B_1 = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

that we found in part (b). For $P_2(\mathbb{C})$ we will use the basis $B_2 = \{z^2, z, 1\}$. Recall that we find the columns of the matrix for H by applying H to the elements of B_1 and finding the coordinates with respect to B_2.

$$H\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right) = (1-1)z^2 + (-1, 0)z + (0+0) = (0)z^2 + (-1)z + (0)1$$

so the first column of our matrix is

$$\left[H\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right) \right]_{B_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The other calculations are

$$H\left(\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right) = (-1)z^2 + (1)z + (1)1$$

and

$$H\left(\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right) = (-i)z^2 + (0)z + (1)1,$$

so the matrix for H is

$$[H]_{B_2, B_1} = \begin{bmatrix} 0 & -1 & -i \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(d) Find the dimension of $\ker(H)$.

Solution: To find the dimension of the kernel, we can row-reduce the matrix found in part (c). The number of non-pivot columns will be the dimension of $\ker(H)$.

$$\begin{bmatrix} 0 & -1 & -i \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF (details omitted)}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there are no non-pivot columns in the reduced matrix, we get

$$\text{nullity}(H) = \dim(\ker(H)) = 0.$$

(e) Find the rank of H.

Solution: The rank-nullity theorem tells us

$$\begin{gathered} \text{rank}(H) + \text{nullity}(H) = \dim(V) \\ \downarrow \\ \text{rank}(H) + 0 = 3 \\ \downarrow \\ \text{rank}(H) = 3. \end{gathered}$$

3. Let V be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each row is equal to 0. Let W be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each column is equal to 0. Find a basis of $V + W$.

Solution: We first want to find bases for V and W. Notice that we can write V as

$$\begin{aligned} V &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : a + b = 0,\ c + d = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : b = -a,\ d = -c \right\} \\ &= \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} : a, c \in \mathbb{C} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, c \in \mathbb{C} \right\}, \end{aligned}$$

so $V = \operatorname{span}(B_1)$ where

$$B_1 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

Since one matrix in B_1 is not a scalar multiple of the other, B_1 is linearly independent, so it is a basis for V. By very similar arguments we get the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

for W.

Now consider the set

$$B_1 \cup B_2 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}.$$

This is not necessarily a basis for $V + W$, but we can use it to find a basis.

Using the standard basis for $M_{2\times 2}(\mathbb{C})$ we can represent a 2×2 matrix by a 4×1 column vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Write each of the matrices in $B_1 \cup B_2$ as 4×1 column vectors in this way, and let them form the columns of a matrix; then apply row-reduction:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF (details omitted)}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three columns of the reduced matrix are pivot columns, while the fourth column is not a pivot column. Therefore, the first three elements of $B_1 \cup B_2$ form a basis for $V + W$. That is,

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$$

is a basis for $V + W$.

4. Let V be a vector space with basis $\mathcal{B}_1 = \{v_1, v_2, \ldots, v_7\}$, and let W be a vector space with basis $\mathcal{B}_2 = \{w_1, w_2, \ldots, w_6\}$. Let $f\colon V \to W$ be the linear map determined by

$$\begin{aligned}
f(v_1) &= w_1 + w_2 - w_4 + 2w_6,\\
f(v_2) &= 3w_1 - w_2 - w_3 + w_5 - 4w_6,\\
f(v_3) &= 2w_2 + 5w_3 - w_4 + 7w_5 - w_6,\\
f(v_4) &= w_1 + w_3 - w_4 + w_6,\\
f(v_5) &= w_2 - 4w_4 + 5w_5 + 3w_6,\\
f(v_6) &= w_1 + w_2 + 2w_3 + 3w_4 + 5w_5,\\
f(v_7) &= 2w_1 - 6w_3 + 2w_4 + w_5 - w_6.
\end{aligned}$$

(a) Write the matrix that represents f relative to bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

Solution: Remember that each $f(v_i)$ gives a column of the matrix. The entries in the matrix are the coefficients of the w_j vectors:

$$[f]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 1 & 2\\ 1 & -1 & 2 & 0 & 1 & 1 & 0\\ 0 & -1 & 5 & 1 & 0 & 2 & -6\\ -1 & 0 & -1 & -1 & -4 & 3 & 2\\ 0 & 1 & 7 & 0 & 5 & 5 & 1\\ 2 & -4 & -1 & 1 & 3 & 0 & -1 \end{bmatrix}.$$

(b) Find the rank and nullity of f.

Solution: We can find the rank of f by row-reducing the matrix $[f]_{\mathcal{B}_2,\mathcal{B}_1}$. The rank will be the number of pivot columns in the reduced matrix. It would be too difficult to reduce the matrix by hand, so we will use MATLAB:

```
A = [ 1  3  0  1  0  1  2  ;
      1 -1  2  0  1  1  0  ;
      0 -1  5  1  0  2 -6  ;
     -1  0 -1 -1 -4  3  2  ;
      0  1  7  0  5  5  1  ;
      2 -4 -1  1  3  0 -1 ];
rref(A)
```

You should get

```
   1.0000         0         0         0         0         0    1.2913
        0    1.0000         0         0         0         0    0.8434
        0         0    1.0000         0         0         0   -0.7987
        0         0         0    1.0000         0         0   -2.4803
        0         0         0         0    1.0000         0    0.4909
        0         0         0         0         0    1.0000    0.6586
```

Since there are six pivot columns in the reduced matrix, f has rank 6. We can then get the nullity from the rank-nullity theorem:

$$\begin{gathered}
\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim(V)\\
\downarrow\\
6 + \operatorname{nullity}(f) = 7\\
\downarrow\\
\operatorname{nullity}(f) = 1.
\end{gathered}$$