Homework 3

Due 02/03/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Consider a map $G : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $G(u) = (x+1)u' - 2u$.
   (a) Show that G is a linear map.
   (b) Find a basis and the dimension of null(G). What is the nullity of G?
   (c) Find a basis and the dimension of range(G). What is the rank of G?
   (d) Is G a monomorphism, epimorphism, isomorphism or none of them?

2. Let
$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : \quad a + b + c + id = 0 \right\}.$$
   Consider a linear map $H : V \to P_2(\mathbb{C})$ given by
$$H\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a+b)z^2 + (b+c)z + (c+d).$$
   (a) Show that V is a subspace of $M_{2\times 2}(\mathbb{C})$.
   (b) Find a basis of V.
   (c) Find a matrix representation of H.
   (d) Find the nullity of H.
   (e) Find the rank of H.
   Hint: use the rank-nullity theorem

3. Let V be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each row is equal to 0. Let W be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each column is equal to 0. Find a basis of $V + W$.

Do the following problem for 6 bonus points.

4. Let V be a vector space with basis $B_1 = \{v_1, v_2, \ldots, v_7\}$, and W be a vector space with basis $B_2 = \{w_1, w_2, \ldots, w_6\}$. Let $f : V \to W$ be a linear map given by
$$\begin{aligned}
f(v_1) &= w_1 + w_2 - w_4 + 2w_6, \\
f(v_2) &= 3w_1 - w_2 - w_3 + w_5 - 4w_6, \\
f(v_3) &= 2w_2 + 5w_3 - w_4 + 7w_5 - w_6, \\
f(v_4) &= w_1 + w_3 - w_4 + w_6, \\
f(v_5) &= w_2 - 4w_4 + 5w_5 + 3w_6, \\
f(v_6) &= w_1 + w_2 + 2w_3 + 3w_4 + 5w_5, \\
f(v_7) &= 2w_1 - 6w_3 + 2w_4 + w_5 - w_6
\end{aligned}$$
   (a) Write the matrix that represents f relative to bases B_1 and B_2.
   (b) Find the rank and nullity of f. (You are encouraged to use Matlab to do this problem. If you use Matlab, please write down the Matlab commands and the outputs.)