Homework 4
Answer Key

1. Consider $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(f(x)) = x\dfrac{d}{dx}(f(2x+1))$.

   (a) Find $[T]_{\mathcal{B}}$, the matrix of T with respect to the basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathbb{P}_2$.
   Solution: Calculate $[Tv]_{\mathcal{B}}$ for each $v \in \mathcal{B}$ to find the columns of the matrix:

$$T(1) = x\frac{d}{dx}[1] = x(0) = 0 \qquad \to \qquad [T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = x\frac{d}{dx}[2x+1] = x(2) = 2x \qquad \to \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T(x^2) = x\frac{d}{dx}[(2x+1)^2] = x(8x+4) = 4x + 8x^2 \qquad \to \qquad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$$

   Thus

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix}.$$

   (b) Find the eigenvalues of T.
   Solution: Notice that $A = [T]_{\mathcal{B}}$ is a triangular matrix, so the eigenvalues are the entries on the diagonal: $\lambda = 0,\ 2,\ 8$. This can also be seen by finding the characteristic polynomial:

$$\det(A - \lambda I) = \begin{bmatrix} 0-\lambda & 0 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 8-\lambda \end{bmatrix} = -\lambda(2-\lambda)(8-\lambda).$$

   Setting the characteristic polynomial equal to zero gives

$$-\lambda(2-\lambda)(8-\lambda) = 0 \qquad \to \qquad \lambda = 0,\ 2,\ 8.$$

   (c) For each eigenvalue, find the eigenvectors associated to the eigenvalue (as polynomials in $\mathbb{P}_2$).
   Solution: To find the eigenvectors associated to λ, we must find $E_\lambda = \text{null}(T - \lambda I)$. This means solving the equation

$$(A - \lambda I)v = 0.$$

   where $v = [v_1, v_2, v_3]^T$. We then must convert the column vectors v back into polynomials. Note that eigenvectors are nonzero, so the eigenvectors associated to the eigenvalue λ are all of the nonzero elements of E_λ.

- First let $\lambda = 0$. Then $A - \lambda I = A$, so we get the augmented system

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The reduced matrix tells us that $v_2 = 0$ and $v_3 = 0$, so

$$v = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_1 \in \mathbb{R}.$$

Converting back to polynomials gives

$$E_0 = \{a : a \in \mathbb{R}\} = \text{span}(\{1\})$$

(here we used $a \in \mathbb{R}$ in place of $v_1 \in \mathbb{R}$).

- Now let $\lambda = 2$. The augemnted system is

$$\left[\begin{array}{ccc|c} 0-2 & 0 & 0 & 0 \\ 0 & 2-2 & 4 & 0 \\ 0 & 0 & 8-2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_1 = 0$ and $v_3 = 0$. Therefore the eigenvalues of A associated to $\lambda = 2$ are

$$v = \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 \in \mathbb{R}.$$

Converting back to polynomials gives

$$E_2 = \{bx : b \in \mathbb{R}\} = \text{span}(\{x\}).$$

- Finally let $\lambda = 8$. The augemnted system is

$$\left[\begin{array}{ccc|c} -8 & 0 & 0 & 0 \\ 0 & -6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_1 = 0$ and $v_2 - \frac{2}{3}v_3 = 0$. Therefore $v_2 = \frac{2}{3}v_3$, so the eigenvalues of A associated to $\lambda = 8$ are

$$v = \begin{bmatrix} 0 \\ \frac{2}{3}v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ \frac{2}{3} \\ 1 \end{bmatrix}, \qquad v_3 \in \mathbb{R}.$$

If we let $c \in \mathbb{R}$ be such that $v_3 = 3c$, then we can write this as

$$c \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \qquad c \in \mathbb{R}.$$

Converting back to polynomials gives

$$E_8 = \{c(2x + 3x^2) : c \in \mathbb{R}\} = \text{span}(\{2x + 3x^2\}).$$

(d) Find a basis $\mathcal{B}'$ of $\mathbb{P}_2$ such that $[T]_{\mathcal{B}'}$ is a diagonal matrix (if possible).

Solution: T is diagonalizable, because the algebraic multiplicity of each eigenvalue matches its geometric multiplicity. We get the basis $\mathcal{B}'$ from the bases for the eigenspaces that we found in part (c):

$$\mathcal{B}' = \{1, x, 2x + 3x^2\}.$$

Then $[T]_{\mathcal{B}'}$ is the diagonal matrix with entries given by the eigenvalues of T:

$$[T]_{\mathcal{B}'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Notice that the eigenvalues along the diagonal are in the same order as their associated eigenvectors in $\mathcal{B}'$.

2. Let V be real vector space and let S and T be linear maps from V to V. Suppose S is invertible.

(a) Prove $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of STS^{-1}.

Solution: This is an if and only if statement, so we need to prove two directions.

- ($\Rightarrow$) Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of T. Then there exists some eigenvector $v \in V$ such that

$$Tv = \lambda v.$$

Now let $w = Sv$ (note that w is nonzero, since v is nonzero and S is invertible). Then

$$\begin{aligned} STS^{-1}w &= STS^{-1}Sv \\ &= STv \\ &= S\lambda v \\ &= \lambda Sv \\ &= \lambda w, \end{aligned}$$

so w is an eigenvector of STS^{-1} with eigenvalue λ. Therefore λ is an eigenvalue of STS^{-1}.

- ($\Leftarrow$) Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of STS^{-1}. Then there exists some eigenvector $w \in V$ such that

$$STS^{-1}w = \lambda w.$$

We can apply S^{-1} to both sides of this equation to get

$$\begin{gathered} S^{-1}STS^{-1}w = S^{-1}\lambda w. \\ \downarrow \\ TS^{-1}w = \lambda S^{-1}w. \end{gathered}$$

Therefore $v = S^{-1}w$ is an eigenvector of T with eigenvalue λ (note that v is nonzero, since w is nonzero and S^{-1} is invertible).

(b) Give a description of the set of eigenvectors of STS^{-1} associated to an eigenvalue λ in terms of the eigenvectors of T associated to λ.

Solution: It can be seen from our work in part (a) that w is an eigenvector of STS^{-1} if and only if $v = S^{-1}w$ is an eigenvector of T. Therefore, the eigenvectors w of STS^{-1} can be written as $w = Sv$. That is, the eigenvectors of STS^{-1} are the image under S of the eigenvectors of T.

3. Two matrices $A, B \in M_{n\times n}(F)$ are said to be *similar* if there is an invertible matrix P such that $A = PBP^{-1}$.

   (a) Is it true that two matrices that are similar to each other must have the same set of eigenvalues? Explain your answer.

   Solution: Yes. Suppose A and B are similar, so $A = PBP^{-1}$. Then asking whether A and B have the same eigenvalues is the same as asking whether PBP^{-1} and B have the same eigenvalues. This is true from problem 2(a) (just replace P with S and B with T).

   (b) Is it true that two matrices that have the same set of eigenvalues must be similar to each other? Explain your answer.

   Solution: No. To prove this is false, we need a counterexample. Let $A, B \in M_{2\times 2}(\mathbb{R})$ be given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad B = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Notice that both matrices have only the eigenvalue 1. Now suppose that we could write $A = PBP^{-1}$ for some invertible matrix P. Then

$$\begin{aligned} A &= PBP^{-1} \\ &= PIP^{-1} \\ &= PP^{-1} \\ &= I, \end{aligned}$$

   but this is a contradiction since $A \neq I$. Therefore A cannot be written as PBP^{-1}, so A and B are not similar, despite having the same eigenvalues.

4. The *trace* of a square matrix is defined as the sum of the entries on the diagonal. Let $A, B \in M_{n\times n}(F)$. Show that $\text{trace}(AB) = \text{trace}(BA)$. Use this result to show that similar matrices have the same trace.

Solution: We will need to use the fact that the (i, i) entry of AB is

$$(AB)_{i,i} = \sum_{k=1}^{n} a_{ik} b_{ki}.$$

The trace of AB is the sum of the (i, i) entries of AB, so

$$\text{trace}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}.$$

Likewise, the trace of BA is

$$\begin{aligned}\text{trace}(BA) &= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki} b_{ik}.\end{aligned}$$

Notice that this last sum for $\text{trace}(BA)$ is the same as the sum for $\text{trace}(AB)$ but with the letters i and k switched. Since i and k are only indices of the sums, we can rename them. That is, we can change i to k and k to i. We then get

$$\text{trace}(BA) = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki} b_{ik} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \text{trace}(AB).$$

Now we want to show that similar matrices have the same trace. Let $A, B \in M_{n\times n}(F)$ be similar matrices, so $A = PBP^{-1}$ for some invertible matrix P. Then

$$\begin{aligned}\text{trace}(A) &= \text{trace}(PBP^{-1}) \\ &= \text{trace}(P(BP^{-1})) \\ &= \text{trace}((BP^{-1})P) \\ &= \text{trace}(BP^{-1}P) \\ &= \text{trace}(BI) \\ &= \text{trace}(B).\end{aligned}$$

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

Solution: We will use MATLAB for this problem. Use the commands

```
A = [ 1  2  3  4  ;
      2  4  6  8  ;
      3  6  9 12  ;
      4  8 12 16 ];
eig(A)
```

You should get

```
ans =

   -0.0000
         0
    0.0000
   30.0000
```

so the eigenvalues of A are 0 and 30. We can also see that the eigenvalue 0 has algebraic multiplicity 3 and the eigenvalue 30 has algebraic multiplicity 1.

(b) For each eigenvalue λ of A, find a basis of the eigenspace $E_\lambda = \text{Null}(A - \lambda I_4)$.

Solution: We will use MATLAB to calculate $E_\lambda = \text{null}(A - \lambda I)$.

- Let $\lambda = 0$. Use the commands

```
B = [ 1  2  3  4  ;
      2  4  6  8  ;
      3  6  9 12  ;
      4  8 12 16 ];
null(B,'r')
```

You should get

```
ans =

    -2    -3    -4
     1     0     0
     0     1     0
     0     0     1
```

Therefore a basis for the eigenspace is

$$B_0 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This shows us that the eigenvalue 0 has geometric multiplicity 3.

- Let $\lambda = 30$. Use the commands

```
B = [ -29   2   3   4  ;
        2 -26   6   8  ;
        3   6 -21  12  ;
        4   8  12 -14 ];
null(B,'r')
```

You should get

```
ans =

    1
    2
    3
    4
```

Therefore a basis for the eigenspace is

$$B_{30} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

This shows us that the eigenvalue 30 has geometric multiplicity 1.

(c) Is A diagonalizable? If so, find a diagonal matrix D and an invertible matrix Q such that $A = QDQ^{-1}$.

Solution: Yes, A is diagonalizable. Notice that for each eigenvalue the geometric multiplicity matches the algebraic multiplicity. The diagonal matrix D is the matrix whose diagonal entries are the eigenvalues of A:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 \end{bmatrix}.$$

Notice that the number of times an eigenvalue occurs on the diagonal matches the multiplicity of the eigenvalue. The matrix Q is formed by the basis vectors for the eigenspaces as found in part (b):

$$Q = \begin{bmatrix} -2 & -3 & -4 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Notice that the order of the eigenvectors in Q matches the order of the eigenvalues along the diagonal in D. That is, the first three columns in Q are the eigenvectors with eigenvalue 0, just as the first three eigenvalues along the diagonal of D are 0.