Homework 4

Due 02/14/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Consider a linear map $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(u)(x) = x\frac{d}{dx}(u(2x+1))$.
   (a) Find $[T]_{\mathcal{B}}$, the matrix of T with respect to the basis $\mathcal{B} = \{1, x, x^2\}$ of $P_2(\mathbb{R})$.
   (b) Find the eigenvalues of T.
   (c) For each eigenvalue, find the eigenvectors associated to the eigenvalue (as polynomials in $P_2(\mathbb{R})$).
   (d) Find a basis $\mathcal{B}'$ of $P_2(\mathbb{R})$ such that $[T]_{\mathcal{B}'}$ is a diagonal matrix (if possible).

2. Let V be real vector space (i.e. a vector space over $\mathbb{R}$). Let S and T be two linear maps from V to V. Suppose S be invertible.
   (a) Show that $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $S \circ T \circ S^{-1}$.
   (b) Give a description of the set of eigenvectors of STS^{-1} associated to an eigenvalue λ in terms of the eigenvectors of T associated to λ.

3. Two matrices $A, B \in M_{n\times n}(F)$ are said to be *similar* if there is an invertible matrix P such that $A = PBP^{-1}$.
   (a) Is it true that two matrices that are similar to each other must have the same set of eigenvalues? Explain your answer.
   (b) Is it true that two matrices that have the same set of eigenvalues must be similar to each other? Explain your answer.

4. The *trace* of a square matrix is defined as the sum of the entries on the diagonal. Let $A, B \in M_{2\times 2}(F)$, where F is a field of numbers. Show that $\text{trace}(AB) = \text{trace}(BA)$. Then use this result to show that similar matrices have the same trace.

Do the following problem for 6 bonus points. You are encouraged to use Matlab to do it. Make sure to write the commands you use and the outputs on your homework.

5. Let
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$
   (a) Find the eigenvalues of A.
   (b) For each eigenvalue λ of A, find a basis of the eigenspace $E_\lambda = \text{null}(A - \lambda I_4)$.
   (c) Is A diagonalizable? If so, find a diagonal matrix D and an invertible matrix Q such that $A = QDQ^{-1}$.