Homework 5
Answer Key

1. Let $f\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a the linear map given by $f(A) = AC - CA$ where

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Is f diagonalizable? If it is, find a basis of $V = M_{2\times 2}(\mathbb{R})$ in which f is represented by a diagonal matrix.

Solution: We start by finding the matrix of f with respect to the basis

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for $M_{2\times 2}(\mathbb{R})$. I will omit the details. The matrix is

$$A = [f]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

Now we find the reduced the characteristic polynomial (I will again omit the details):

$$\det(A - \lambda I) = \lambda^4 - 4\lambda^2 = \lambda^2(2-\lambda)(2+\lambda).$$

Setting the characteristic polynomial equal to zero gives

$$\lambda^2(2-\lambda)(2+\lambda) = 0 \qquad \to \qquad \lambda = 0, 2, -2.$$

We now need to find a basis for each eigenspace.

- $\boldsymbol{\lambda = 0}$**:** We first want to find $\text{null}(A - \lambda I)$. That is, we want to solve $(A - \lambda I)v = 0$ for v. The augmented system for this equation is

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Reduce}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This gives the equations

$$\begin{aligned} v_1 - v_4 &= 0 \\ v_2 - v_3 &= 0 \end{aligned} \qquad \to \qquad \begin{aligned} v_1 &= v_4 \\ v_2 &= v_3. \end{aligned}$$

The eigenvector v is then

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_2 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_1, v_2 \in \mathbb{R}.$$

Notice that the vectors $[1\ 0\ 0\ 1]^T$ and $[0\ 1\ 1\ 0]^T$ are linearly independent, since neither is a scalar multiple of the other.

We now need to convert these column vectors back into 2×2 matrices:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \to \qquad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore a basis for the eigenspace E_0 is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

- $\boldsymbol{\lambda = 2}$**:** The augmented system for the equation $(A - \lambda I)v = 0$ is

$$\left[\begin{array}{rrrr|r} -2 & 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ -1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Reduce}} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This gives the equations

$$\begin{aligned} v_1 + v_4 &= 0 \\ v_2 + v_4 &= 0 \\ v_3 - v_4 &= 0 \end{aligned} \qquad \to \qquad \begin{aligned} v_1 &= -v_4 \\ v_2 &= -v_4 \\ v_3 &= v_4. \end{aligned}$$

The eigenvector v is

$$v = v_4 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad v_4 \in \mathbb{R}.$$

Converting to a 2×2 matrix gives a basis for E_2:

$$\left\{ \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\}.$$

- $\boldsymbol{\lambda = -2}$**:** The augmented system for the equation $(A - \lambda I)v = 0$ is

$$\left[\begin{array}{rrrr|r} 2 & 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{Reduce}} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This gives the equations

$$\begin{aligned} v_1 + v_4 &= 0 \\ v_2 + v_4 &= 0 \\ v_3 - v_4 &= 0 \end{aligned} \qquad \to \qquad \begin{aligned} v_1 &= -v_4 \\ v_2 &= v_4 \\ v_3 &= -v_4. \end{aligned}$$

The eigenvector v is

$$v = v_4 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad v_4 \in \mathbb{R}.$$

Converting to a 2×2 matrix gives a basis for E_2:

$$\left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

For each eigenvalue, the geometric multiplicity matches the algebraic multiplicity, so the matrix is diagonalizable. We form the desired basis for $M_{2\times 2}(\mathbb{R})$ by combining the bases for the eigenspaces:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

The matrix for f with respect to the basis $\mathcal{B}$ is then the diagonal matrix with the eigenvalues along the diagonal:

$$[f]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Notice that the order of the eigenvalues along the diagonal of $[f]_{CB}$ is the same as the order of the associated eigenvectors in the basis $\mathcal{B}$.

2. On $\mathbb{R}^2$ let us consider three operators $(\cdot,\cdot)_1$, $(\cdot,\cdot)_2$, $(\cdot,\cdot)_3$ given by

$$\begin{aligned} (x,y)_1 &= x_1y_1 + 2x_2y_2, \\ (x,y)_2 &= x_1x_2 + y_1y_2, \\ (x,y)_3 &= (x_1+x_2)(y_1+y_2) \end{aligned}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Which of these operators are inner products on $\mathbb{R}^2$? Which are not? Explain your answer with proof or counterexample.

Solution:

- $\boldsymbol{(x,y)_1 = x_1y_1 + 2x_2y_2}$**:** This is an inner product. We will need to prove each property of the inner product. Let

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2, \quad \text{and} \quad \lambda \in \mathbb{R}.$$

  - **Positivity:**

$$(v,v)_1 = (v_1)^2 + 2(v_2)^2 \geq 0 + 0 \geq 0.$$

    since the square of a real number is always greater than or equal to 0.
  - **Definiteness:** First note that

$$(0,0)_1 = 0^2 + 2(0)^2 = 0.$$

    Now suppose

$$(v,v)_1 = (v_1)^2 + 2(v_2)^2 = 0.$$

    Since $(v_1)^2$ and $2(v_2)^2$ are both greater than or equal to zero, the only way to make this equation true is if $(v_1)^2 = 0$ and $2(v_2)^2 = 0$. Solving for v_1 and v_2 gives $v_1 = 0$ and $v_2 = 0$. Therefore $v = (0,0)$.
  - **Additivity in the first component:**

$$\begin{aligned} (u+v,w)_1 &= (u_1+v_1)w_1 + 2(u_2+v_2)w_2 \\ &= u_1w_1 + v_1w_1 + 2u_2w_2 + 2v_2w_2 \\ &= (u_1w_1 + 2u_2w_2) + (v_1w_1 + 2v_2w_2) \\ &= (u,w)_1 + (v,w)_1. \end{aligned}$$

- **Homogeneity in the first component:**

$$\begin{aligned}(\lambda u, v)_1 &= (\lambda u_1)v_1 + 2(\lambda u_2)v_2 \\ &= \lambda(u_1v_1 + 2u_2v_2) \\ &= \lambda(u, v)_1\end{aligned}$$

- **Conjugate symmetry:** Note that for any real number a, the complex conjugate is $\overline{a} = a$. Now

$$\begin{aligned}\overline{(v, u)_1} &= \overline{v_1u_1 + v_2u_2} \\ &= v_1u_1 + v_2u_2 && \text{since } v_1u_1 + v_2u_2 \text{ is a real number} \\ &= u_1v_1 + u_2v_2 \\ &= (u, v)_1.\end{aligned}$$

- $\boldsymbol{(x, y)_2 = x_1x_2 + y_1y_2}$**:** This is not an inner product. In fact, the only property that this operator satisfies is conjugate symmetry. You only need to give a single counterexample, but I will give a counterexample for each property that is not satisfied.
  - **Positivity:**
$$\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)_2 = (1)(-1) + (1)(-1) = -2 < 0$$
  - **Definiteness:**
$$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)_2 = (1)(0) + (1)(0) = 0,$$
  - **Additivity in the first component:**
$$\begin{aligned}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 &= \left(\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 \\ &= (2)(2) + (0)(0) \\ &= 4,\end{aligned}$$

    but

$$\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 + \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 = [(1)(1) + (0)(0)] + [(1)(1) + (0)(0)] = 2.$$
  - **Homogeneity in the first component:**
$$\begin{aligned}\left(2\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 &= \left(\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 \\ &= (2)(2) + (0)(0) = 4,\end{aligned}$$

    but
$$2\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)_2 = 2[(1)(1) + (0)(0)] = 2.$$

- $\boldsymbol{(x, y)_3 = (x_1 + x_2)(y_1 + y_2)}$**:** This is not an inner product. The only property that it does not satisfy is definiteness:

$$\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)_3 = (1-1)(1-1) = (0)(0) = 0$$

3. On $\mathbb{C}^2$ consider the vectors $u_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $(\cdot,\cdot)$ be an inner product on $\mathbb{C}^2$ that satisfies

$$(u_1, u_2) = i, \qquad (u_1, u_1) = 3, \qquad (u_2, u_2) = 1.$$

Compute $\left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$.

Solution: First we need to write the vectors $\begin{bmatrix} i+1 \\ 2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ in terms of u_1 and u_2:

$$\begin{aligned} \begin{bmatrix} i+1 \\ 2i \end{bmatrix} &= (2i)u_1 + (3+i)u_2 \\ \begin{bmatrix} 1 \\ i \end{bmatrix} &= (i)u_1 + (2)u_2. \end{aligned}$$

We can now use linearity in the first component, conjugate linearity in the second component, and conjugate symmetry to calculate:

$$\begin{aligned} \left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) &= \Big((2i)u_1 + (3+i)u_2, (i)u_1 + (2)u_2\Big) \\ &= (2i)\Big(u_1, (i)u_1 + (2)u_2\Big) + (3+i)\Big(u_2, (i)u_1 + (2)u_2\Big) \\ &= (2i)\overline{(i)}\Big(u_1, u_1\Big) + (2i)\overline{(2)}\Big(u_1, u_2\Big) + (3+i)\overline{(i)}\Big(u_2, u_1\Big) + (3+i)\overline{(2)}\Big(u_2, u_2\Big) \\ &= (2i)\overline{(i)}\Big(u_1, u_1\Big) + (2i)\overline{(2)}\Big(u_1, u_2\Big) + (3+i)\overline{(i)}\overline{\Big(u_1, u_2\Big)} + (3+i)\overline{(2)}\Big(u_2, u_2\Big) \\ &= (2i)(-i)(3) + (2i)(2)(i) + (3+i)(-i)(-i) + (3+i)(2)(1) \\ &= (6) + (-4) + (3+i) + (6+2i) \\ &= 11 + 3i. \end{aligned}$$

4. Let V be a vector space of dimension n over $F = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}$ be a basis a V. Consider the operator $(\cdot,\cdot)$ on V given by

$$(u, v) = c_1\bar{d}_1 + c_2\bar{d}_2 + \cdots + c_n\bar{d}_n$$

where

$$[u]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \qquad [v]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

Show that $(\cdot,\cdot)$ is an inner product on V.

Solution: We need to prove each property of an inner product space. Let

$$[u]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \qquad [v]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}, \qquad [w]_{\mathcal{B}} = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix},$$

and let $\lambda \in F$.

- **Positivity:** Recall that any complex number z can be written as $x+iy$ where $x, y \in \mathbb{R}$. We then have

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 \geq 0$$

since the square of any real number is always greater than or equal to 0. Therefore

$$(u,u) = c_1\bar{c}_1 + c_2\bar{c}_2 + \cdots + c_n\bar{c}_n \geq 0$$

- **Definiteness:** This is an if and only if statement, so we need to show both directions.

($\Rightarrow$) We saw in the proof of positivity that,

$$(u,u) = c_1\bar{c}_1 + c_2\bar{c}_2 + \cdots + c_n\bar{c}_n$$

is the sum of real number that are each greater than or equal to 0. If this sum is equal to 0, then each term in the sum must be 0. That is,

$$c_k\bar{c}_k = 0 \quad \text{for every } k = 1, \ldots, n.$$

If we write $c_k = x + iy$ where $x, y \in \mathbb{R}$, then

$$0 = c_k\bar{c}_k = x^2 + y^2,$$

so $x = y = 0$. Therefore $c_k = 0$ for every $k = 1, \ldots, n$, so $u = 0$.

($\Leftarrow$) If $u = 0$, then

$$(u,u) = 0 \cdot \bar{0} + 0 \cdot \bar{0} + \cdots + 0 \cdot \bar{0} = 0.$$

- **Additivity in the first component:**

$$\begin{aligned}
(u+v,w) &= (c_1+d_1)\bar{e}_1 + (c_2+d_2)\bar{e}_2 + \cdots + (c_n+d_n)\bar{e}_n \\
&= c_1\bar{e}_1 + d_1\bar{e}_1 + c_2\bar{e}_2 + d_2\bar{e}_2 + \cdots + c_n\bar{e}_n + d_n\bar{e}_n \\
&= \big(c_1\bar{e}_1 + c_2\bar{e}_2 + \cdots + c_n\bar{e}_n\big) + \big(d_1\bar{e}_1 + d_2\bar{e}_2 + \cdots + d_n\bar{e}_n\big) \\
&= (u,w) + (v,w).
\end{aligned}$$

- **Homogeneity in the first component:**

$$\begin{aligned}
(\lambda u, v) &= (\lambda c_1)\bar{d}_1 + (\lambda c_2)\bar{d}_2 + \cdots + (\lambda c_n)\bar{d}_n \\
&= \lambda(c_1\bar{d}_1 + c_2\bar{d}_2 + \cdots + c_n\bar{d}_n) \\
&= \lambda(u,v)
\end{aligned}$$

- **Conjugate symmetry:**

$$\begin{aligned}
\overline{(v,u)} &= \overline{c_1\bar{d}_1 + c_2\bar{d}_2 + \cdots + c_n\bar{d}_n} \\
&= \bar{c}_1 d_1 + \bar{c}_2 d_2 + \cdots + \bar{c}_n d_n \\
&= d_1\bar{c}_1 + d_2\bar{c}_2 + \cdots + d_n\bar{c}_n \\
&= (u,v).
\end{aligned}$$

5. On $\mathbb{R}^2$, consider two vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Define an inner product on $\mathbb{R}^2$ so that v_1 and v_2 are orthogonal to each other (i.e. having inner product equal to zero).

Solution: The easiest way to do this is probably to follow something similar to what was done in problem 4. Notice that

$$\mathcal{B} = \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, because neither vector is a scalar multiple of the other. Since $\mathbb{R}^2$ has dimension 2, this means that $\mathcal{B}$ is a basis for $\mathbb{R}^2$. Now we can define our inner product $(\cdot, \cdot)$ in a very similar way to how the inner product was defined in problem 4:

$$(u, v) = c_1 d_1 + c_2 d_2$$

where

$$[u]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad [v]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

Since v_1 and v_2 are the elements of $\mathcal{B}$, they can easily be written in $\mathcal{B}$ coordinates:

$$[v_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [v_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$(v_1, v_2) = (1)(0) + (0)(1) = 0,$$

so v_1 and v_2 are orthogonal to each other, as desired.

We now want to show that $(\cdot, \cdot)$ is an inner product, so we need to prove each inner product property. Let

$$[u]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad [v]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \qquad [w]_{\mathcal{B}} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

and let $\lambda \in \mathbb{R}$.

- **Positivity:**

$$(u, u) = (c_1)^2 + (c_2)^2 \geq 0$$

  since the square of any real number is always greater than or equal to 0.

- **Definiteness:** This is an if and only if statement, so we need to show both directions.

  ($\Rightarrow$) since the square of any real number is always greater than or equal to 0, the only way to get

$$(u, u) = (c_1)^2 + (c_2)^2 = 0$$

  is if $(c_1)^2 = 0$ and $(c_2)^2 = 0$. This implies that $c_1 = 0$ and $c_2 = 0$, so $u = 0$.

  ($\Leftarrow$) If $u = 0$, then

$$[u]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

  so

$$(u, u) = 0 \cdot 0 + 0 \cdot 0 = 0.$$

- **Additivity in the first component:**

$$\begin{aligned}(u+v,w) &= (c_1+d_1)e_1 + (c_2+d_2)e_2 \\ &= c_1e_1 + d_1e_1 + c_2e_2 + d_2e_2 \\ &= \big(c_1e_1 + c_2e_2\big) + \big(d_1e_1 + d_2e_2\big) \\ &= (u,w) + (v,w).\end{aligned}$$

- **Homogeneity in the first component:**

$$\begin{aligned}(\lambda u, v) &= (\lambda c_1)d_1 + (\lambda c_2)d_2 \\ &= \lambda(c_1d_1 + c_2d_2) \\ &= \lambda(u,v)\end{aligned}$$

- **Conjugate symmetry:**

$$\begin{aligned}\overline{(v,u)} &= \overline{c_1d_1 + c_2d_2} \\ &= c_1d_1 + c_2d_2 && \text{since } c_1d_1 + c_2d_2 \text{ is a real number} \\ &= d_1c_1 + d_2c_2 \\ &= (u,v).\end{aligned}$$