Homework 5

Due 02/21/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Let $f : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear map given by $f(A) = AC - CA$ where
$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
Is f diagonalizable? If it is, find a basis of $V = M_{2\times 2}(\mathbb{R})$ in which f is represented by a diagonal matrix.

2. On $\mathbb{R}^2$, let us consider three operators $(\cdot,\cdot)_1$, $(\cdot,\cdot)_2$, $(\cdot,\cdot)_3$ given by
$$\begin{aligned} (x,y)_1 &= x_1y_1 + 2x_2y_2, \\ (x,y)_2 &= x_1x_2 + y_1y_2, \\ (x,y)_3 &= (x_1+x_2)(y_1+y_2) \end{aligned}$$
where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Which of these operators are inner products on $\mathbb{R}^2$? Which are not? Explain your answer with proof or counterexample.

3. On $\mathbb{C}^2$, consider two vectors $u_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $(\cdot,\cdot)$ be an inner product that satisfies
$$(u_1,u_2) = i, \quad (u_1,u_1) = 3, \quad (u_2,u_2) = 1.$$
Compute $\left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$.

4. Let V be a vector space over $F = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}$ be a basis a V. Consider an operator on V given by
$$(u,v) = c_1\bar{d}_1 + c_2\bar{d}_2 + \ldots + c_n\bar{d}_n,$$
where
$$[u]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \qquad [v]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$
Show that $(\cdot,\cdot)$ is an inner product on V.

Do the following problem for 6 bonus points.

5. On $\mathbb{R}^2$, consider two vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Define an inner product on $\mathbb{R}^2$ so that v_1 and v_2 are perpendicular to each other (i.e. having inner product equal to zero).