Homework 6
Answer Key

1. Let V be an inner product space with inner product $(\cdot,\cdot)$ and let $\|\cdot\|\colon V \to \mathbb{R}$ be given by $\|v\| = \sqrt{(v,v)}$. Show that

   (a) (Parallelogram Law)

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \qquad \forall u, v \in V.$$

Solution: Let $u, v \in V$. Then

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= (u+v, u+v) + (u-v, u-v) \\
&= (u+v, u+v) + (u,u) - (u,v) - (v,u) + (v,v) \\
&= (u,u) + (u,v) + (v,u) + (v,v) + (u,u) - (u,v) - (v,u) + (v,v) \\
&= 2(u,u) + 2(v,v) \\
&= 2\|u\|^2 + 2\|v\|^2 \\
&= 2\left(\|u\|^2 + \|v\|^2\right).
\end{aligned}$$

   (b) (Cauchy-Schwartz inequality)

$$|(u,v)| \le \|u\|\|v\| \qquad \forall u, v \in V.$$

Solution: Before we begin the proof, recall that

$$\alpha\overline{\alpha} = |\alpha|^2$$

for any $\alpha \in \mathbb{C}$. We now proceed with the proof.
Let $u, v \in V$ and let $F \subseteq \mathbb{C}$ be the field of scalars associated to the vector space V. For any $\alpha \in F$ we have by positivity of the inner product that

$$0 \le (\alpha u + v, \alpha u + v).$$

Expanding the right-hand side gives

$$\begin{aligned}
0 &\le (\alpha u + v, \alpha u + v) \\
&= \alpha(u, \alpha u + v) + (v, \alpha u + v) && \text{(linearity in the first component)} \\
&= \alpha\big(\overline{\alpha}(u,u) + (u,v)\big) + \overline{\alpha}(v,u) + (v,v) && \text{(conjugate linearity in the second component)} \\
&= \alpha\overline{\alpha}(u,u) + \alpha(u,v) + \overline{\alpha}(v,u) + (v,v) \\
&= \alpha\overline{\alpha}\|u\|^2 + \alpha(u,v) + \overline{\alpha}(v,u) + \|v\|^2 \\
&= \alpha\overline{\alpha}\|u\|^2 + \alpha(u,v) + \overline{\alpha}\overline{(u,v)} + \|v\|^2
\end{aligned}$$

Since this is true for all $\alpha \in F$, it must be true for

$$\alpha = -\frac{\overline{(u,v)}}{\|u\|^2}.$$

Plugging this into our inequality gives

$$
\begin{aligned}
0 &\le \alpha\overline{\alpha}\|u\|^2 + \alpha(u,v) + \overline{\alpha}\overline{(u,v)} + \|v\|^2 \\
&= \left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)\overline{\left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)}\|u\|^2 + \left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)(u,v) + \overline{\left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)}\overline{(u,v)} + \|v\|^2 \\
&= \left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)\left(-\frac{(u,v)}{\|u\|^2}\right)\|u\|^2 + \left(-\frac{\overline{(u,v)}}{\|u\|^2}\right)(u,v) + \left(-\frac{(u,v)}{\|u\|^2}\right)\overline{(u,v)} + \|v\|^2 \\
&= \left(\frac{\overline{(u,v)}(u,v)}{\|u\|^2}\right) - \left(\frac{\overline{(u,v)}(u,v)}{\|u\|^2}\right) - \left(\frac{(u,v)\overline{(u,v)}}{\|u\|^2}\right) + \|v\|^2 \\
&= -\left(\frac{(u,v)\overline{(u,v)}}{\|u\|^2}\right) + \|v\|^2 \\
&= -\frac{|(u,v)|^2}{\|u\|^2} + \|v\|^2.
\end{aligned}
$$

We now add $\frac{|(u,v)|^2}{\|u\|^2}$ to both sides of the inequality to get

$$\frac{|(u,v)|^2}{\|u\|^2} \le \|v\|^2.$$

Multiplying by $\|u\|^2$ gives

$$|(u,v)|^2 \le \|u\|^2\|v\|^2.$$

Taking the square root of both sides gives the desired inequality:

$$|(u,v)| \le \|u\|\|v\|.$$

2. On $\mathbb{R}^n$ consider the operator $\|\cdot\|_1$ defined by

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{k=1}^{n} |x_k|$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Show that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$. (This is known as a "taxicab" norm.)

Solution: We need to prove the four properties of a norm. Let $x, y \in \mathbb{R}^n$ where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

and let $\lambda \in \mathbb{R}$.

- **Positivity:**

$$\begin{aligned}\|x\|_1 &= \sum_{k=1}^{n} |x_k| \\ &\geq \sum_{k=1}^{n} 0 && \text{(since } |x_k| \geq 0 \text{ for each } k = 1, \ldots, n\text{)} \\ &= 0.\end{aligned}$$

- **Definiteness:** This is an if and only if statement, so we need to prove both directions:
  - If $x = 0$, then
  $$\|x\|_1 = \sum_{k=1}^{n} 0 = 0.$$
  - Suppose $\|x\|_1 = 0$. Then
  $$\sum_{k=1}^{n} |x_k| = \|x\|_1 = 0.$$
  Since each $|x_k|$ is greater than or equal to 0, the only way for the above equation to hold is if $|x_k| = 0$ for all $k = 1, \ldots, n$. Since $|x_k| = 0$ implies $x_k = 0$, we have that every component of x is 0, so $x = 0$.

- **Homogeneity:**

$$\begin{aligned}\|\lambda x\|_1 &= \sum_{k=1}^{n} |\lambda x_k| \\ &= \sum_{k=1}^{n} |\lambda||x_k| && \text{(by homogeneity of the absolute value)} \\ &= |\lambda| \sum_{k=1}^{n} |x_k| && \text{(since } \lambda \text{ is a constant not depending on } k\text{)} \\ &= |\lambda| \|x\|_1.\end{aligned}$$

- **Triangle inequality:**

$$\begin{aligned}\|x + y\|_1 &= \sum_{k=1}^{n} |x_k + y_k| \\ &\leq \sum_{k=1}^{n} \big(|x_k| + |y_k|\big) && \text{(since the absolute value satisfies the triangle inequality)} \\ &= \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k| \\ &= \|x\|_1 + \|y\|_1.\end{aligned}$$

3. Let V be an inner product space with inner product $(\cdot, \cdot)$ and let $S \subseteq V$. The *orthogonal complement* of S is defined as

$$S^\perp = \{v \in V \mid (s, v) = 0 \ \ \forall s \in S\}.$$

Let U be a finite dimensional subspace of V.

(a) Show that $U^\perp$ is a subspace of V.

Solution: We need to prove the three subspace axioms:

- ***$U^\perp$* contains 0:** Recall that $(u, 0) = 0$ for any $u \in V$. Then $(u, 0) = 0$ for any $u \in U$, so $0 \in U^\perp$.
- ***$U^\perp$* is closed under addition:** Let $v, w \in U^\perp$. Then for any $u \in U$ we have

$$\begin{aligned}(u, v + w) &= (u, v) + (u, w) \\ &= 0 + 0 && \text{(since } u \in U \text{ and } v, w \in U^\perp\text{)} \\ &= 0.\end{aligned}$$

  Therefore $v + w \in U^\perp$.
- ***$U^\perp$* is closed under scalar multiplication:** Let $v \in U^\perp$ and let λ be a scalar. Then for any $u \in U$ we have

$$\begin{aligned}(u, \lambda v) &= \overline{\lambda}(u, v) && \text{(conjugate linearity in the second component)} \\ &= \overline{\lambda}(0) && \text{(since } u \in U \text{ and } v \in U^\perp\text{)} \\ &= 0.\end{aligned}$$

  Therefore $\lambda v \in U^\perp$.

Note that we did not need to use the fact that U is finite-dimensional.

(b) Show that $U \oplus U^\perp = V$.

Solution: We first want to show that $V = U + U^\perp$. Since U and $U^\perp$ are subsets of V, we have that $U + U^\perp \subseteq V$. We now need to show $V \subseteq U + U^\perp$.

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be an orthonormal basis for U. That is, $\mathcal{B}$ is a basis for U in which

- $(e_j, e_k) = 0$ whenever $j \neq k$,
- $\|e_k\| = 1$ for $k = 1, \ldots, n$.

Such a basis exists by the Gram-Schmidt process. Now define the map $P\colon V \to V$ by

$$P(v) = \sum_{k=1}^{n} (v, e_k) e_k.$$

Let v be an arbitrary element of V. Then

$$v = P(v) + (v - P(v)).$$

Notice that $P(v)$ is always an element of U, since it is a linear combination of the elements in the basis $\mathcal{B}$ for U. To prove that $v \in U + U^\perp$, we want to show that $(v - P(v)) \in U^\perp$.

Consider an arbitrary element e_j of the basis $\mathcal{B}$ for U. We then have

$$\begin{aligned}
(e_j, v - P(v)) &= (e_j, v) - (e_j, P(v)) \\
&= (e_j, v) - \left(e_j, \sum_{k=1}^{n} (v, e_k) e_k \right) \\
&= (e_j, v) - \sum_{k=1}^{n} \overline{(v, e_k)} (e_j, e_k) && \left(\begin{array}{l} \text{conjugate linearity in} \\ \text{the second component} \end{array} \right) \\
&= (e_j, v) - \overline{(v, e_j)} (e_j, e_j) && (\text{since } (e_j, e_k) = 0 \text{ whenever } j \neq k) \\
&= (e_j, v) - \overline{(v, e_j)} \|e_j\|^2 \\
&= (e_j, v) - (e_j, v) \|e_j\|^2 \\
&= (e_j, v) - (e_j, v) && (\text{since } \|e_j\| = 1) \\
&= 0.
\end{aligned}$$

Now any element $u \in U$ can be written as a linear combination of elements in $\mathcal{B}$, so

$$u = \sum_{k=1}^{n} \alpha_k e_k, \qquad \text{where } \alpha_1, \ldots, \alpha_n \text{ are scalars.}$$

Then

$$\begin{aligned}
(u, v - P(v)) &= \left(\sum_{k=1}^{n} \alpha_k e_k, v - P(v) \right) \\
&= \sum_{k=1}^{n} \alpha_k (e_k, v - P(v)) \\
&= \sum_{k=1}^{n} \alpha_k (0) \\
&= 0,
\end{aligned}$$

so $(v - P(v)) \in U^\perp$. Therefore

$$u = P(v) + \big(v - P(v)\big) \in U + U^\perp,$$

so $V \subseteq U + U^\perp$ and hence $V = U + U^\perp$.

We now want to show that this sum is direct. That is, we want to prove $U \cap U^\perp = \{0\}$. Let $v \in U \cap U^\perp$. Since $v \in U^\perp$ we have

$$(u, v) = 0 \quad \text{for all } u \in U.$$

Since v is also in U, we can set $u = v$ to get

$$(v, v) = 0.$$

By definiteness of the inner product, we must have $v = 0$. Since v was an arbitrary element of $U \cap U^\perp$, we get $U \cap U^\perp = \{0\}$ and hence $U \oplus U^\perp = V$.

(c) Show that $(U^\perp)^\perp = U$.

Solution: First, we want to show that $U \subseteq (U^\perp)^\perp$. Let $v \in U$. Then for every $w \in U^\perp$ we have

$$\begin{aligned} (w, v) &= \overline{(v, w)} \\ &= \overline{0} && \text{since } v \in U \text{ and } w \in U^\perp \\ &= 0. \end{aligned}$$

Therefore $v \in (U^\perp)^\perp$, so $U \subseteq (U^\perp)^\perp$.

We now want to show that $(U^\perp)^\perp \subseteq U$. Let $v \in (U^\perp)^\perp$. Since $v \in V$ it can be written as

$$v = u + w \qquad u \in U,\ w \in U^\perp$$

by problem 2(b). We want to show that $w = 0$, so that $v = u \in U$. Notice that since $w \in U^\perp$ and $v \in (U^\perp)^\perp$ we get

$$(v, w) = \overline{(w, v)} = \overline{0} = 0.$$

Then

$$\begin{aligned} 0 &= (v, w) \\ &= (u + w, w) \\ &= (u, w) + (w, w) \\ &= 0 + (w, w) && (\text{since } u \in U \text{ and } w \in U^\perp) \\ &= (w, w), \end{aligned}$$

so $w = 0$ by the definiteness of the inner product. Therefore $v = u \in U$, so $(U^\perp)^\perp \subseteq U$ and hence $(U^\perp)^\perp = U$.

4. Let V be an inner product space over $\mathbb{C}$ with an orthonormal basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$. Show that if $x = \sum_{k=1}^{n} \alpha_k v_k$ and $y = \sum_{k=1}^{n} \beta_k v_k$ where $\alpha_k, \beta_k \in \mathbb{C}$ for $k = 1, 2, ..., n$ then

$$(x, y) = \sum_{k=1}^{n} \alpha_k \overline{\beta_k}.$$

Solution: Recall that to be orthonormal, the elements of $\mathcal{B}$ must satisfy

- $(v_j, v_k) = 0$ whenever $j \neq k$,
- $\|v_k\| = 1$ for $k = 1, \ldots, n$.

Now

$$\begin{aligned}
(x, y) &= \left(\sum_{j=1}^{n} \alpha_j v_j, \sum_{k=1}^{n} \beta_k v_k \right) \\
&= \sum_{j=1}^{n} \alpha_j \left(v_j, \sum_{k=1}^{n} \beta_k v_k \right) && \text{(linearity in the first component)} \\
&= \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{n} \overline{\beta_k} \, (v_j, v_k) && \text{(conjugate linearity in the second component)} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\beta_k} (v_j, v_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\beta_k} (v_j, v_k) \\
&= \sum_{k=1}^{n} \alpha_k \overline{\beta_k} (v_k, v_k) && \text{(since } (v_j, v_k) = 0 \text{ whenever } j \neq k) \\
&= \sum_{k=1}^{n} \alpha_k \overline{\beta_k} \|v_k\|^2 \\
&= \sum_{k=1}^{n} \alpha_k \overline{\beta_k}.
\end{aligned}$$

5. Let $P_n(\mathbb{R})$ (where $n \geq 3$) be equipped with the inner product

$$(f(x), g(x)) = \int_0^1 f(x)g(x)\,dx.$$

Find $\|x^2 - p\|$ (using the norm induced by the inner product).

Solution: First recall that the projection of f onto g is defined as

$$\text{proj}_g(f) = \frac{(f,g)}{(g,g)}g$$

First we need to calculate p. To do this we need to calculate two inner products:

$$\begin{aligned}
(x^2, x) &= \int_0^1 (x^2)(x)\,dx &= \int_0^1 x^3\,dx &= \tfrac{1}{4}x^4\Big|_0^1 &= \tfrac{1}{4}(1)^4 - \tfrac{1}{4}(0)^4 &= \frac{1}{4}, \\
(x, x) &= \int_0^1 (x)(x)\,dx &= \int_0^1 x^2\,dx &= \tfrac{1}{3}x^3\Big|_0^1 &= \tfrac{1}{3}(1)^3 - \tfrac{1}{3}(0)^3 &= \frac{1}{3}.
\end{aligned}$$

We can now calculate p:

$$p = \text{proj}_x\left(x^2\right) = \frac{(x^2, x)}{(x,x)}x = \frac{\frac{1}{4}}{\frac{1}{3}}x = \tfrac{3}{4}x.$$

Now we want to calculate $\|x^2 - p\|$. Recall that the norm associated to the inner product is defined by $\|f\| = \sqrt{(f,f)}$. Therefore

$$\begin{aligned}
(x^2 - p, x^2 - p) &= (x^2 - \tfrac{3}{4}x, x^2 - \tfrac{3}{4}x) \\
&= \int_0^1 (x^2 - \tfrac{3}{4}x)(x^2 - \tfrac{3}{4}x)\,dx \\
&= \int_0^1 x^4 - \tfrac{3}{2}x^3 + \frac{9}{16}x^2\,dx \\
&= \left(\tfrac{1}{5}x^5 - \tfrac{3}{8}x^4 + \tfrac{3}{16}x^2\right)\Big|_0^1 \\
&= \left(\tfrac{1}{5} - \tfrac{3}{8} + \tfrac{3}{16}\right) - (0) \\
&= \frac{1}{80},
\end{aligned}$$

so

$$\|x^2 - p\| = \sqrt{(x^2 - p, x^2 - p)} = \sqrt{\frac{1}{80}} = \frac{1}{\sqrt{80}} = \frac{1}{4\sqrt{5}}.$$