Homework 6

Due 02/28/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

1. Let V be an inner product space and let $\|\cdot\|$ be the norm associated with the inner product on V, i.e. given by $||v|| = \sqrt{(v,v)}$. Show that

   (a) (Parallelogram identity)
   $$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \qquad \forall\, u, v \in V.$$

   (b) (Cauchy-Schwarz inequality)
   $$|(u,v)| \le \|u\|\,\|v\| \qquad \forall\, u, v \in V.$$

   Note: for Part (b), you only need to show proof for the case $F = \mathbb{R}$ or $\mathbb{Q}$. Use the fact that $(tu+v, tu+v) \ge 0$ for all $t \in \mathbb{R}$. If you can do the case $F = \mathbb{C}$, you will be granted 3 extra points.

2. On $\mathbb{R}^n$, let us consider an operator $\|\cdot\|$ given by $\|x\| = |x_1| + |x_2| + \ldots + |x_n|$ where
   $$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$
   Show that $\|\cdot\|$ is a norm on $\mathbb{R}^n$. (This is known as a "taxicab" norm.)

3. Let V be an inner product space. For each subset $S \subset V$, the *orthogonal complement* of S is denoted by $S^\perp = \{v \in V :\ v \perp w \text{ for all } w \in S\}$. Let U be a subspace of V. Show the following statements:

   (a) $U^\perp$ is a subspace of V.

   (b) $U \oplus U^\perp = V$.

   (c) $(U^\perp)^\perp = U$.

4. Let V be an inner product space over $\mathbb{C}$ with an orthonormal basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$. Let $x = \sum_{k=1}^n \alpha_k v_k$ and $y = \sum_{k=1}^n \beta_k v_k$ where $\alpha_k, \beta_k \in \mathbb{C}$ for $k = 1, 2, ..., n$. Show that
   $$(x,y) = \sum_{k=1}^n \alpha_k \bar{\beta}_k.$$

Do the following problem for 6 bonus points.

5. On $P_n(\mathbb{R})$, which is the space of polynomials of real coefficients with degree $\le n$, let us consider the inner product
   $$(f,g) = \int_0^1 f(x)g(x)dx.$$
   Let p be the orthogonal projection of x^2 on the line span$\{x\}$. Determine $\|x^2 - p\|$.