Homework 7

Answer Key

Assume the inner products on $\mathbb{R}^n$ and $M_{m\times n}(\mathbb{R})$ are respectively:

$$\begin{aligned}(x,y) &= x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{j=1}^{n} x_jy_j,\\ (A,B) &= a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{mn}b_{mn} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}b_{jk}.\end{aligned}$$

1. Consider the vector space $E = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_2 = x_1 + x_3 + x_4 \right\}$.

   (a) Find an orthonormal basis of E.

   Solution: First we want to find a basis for E that is not necessarily orthonormal. We can rewrite E as

$$\begin{aligned} E &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_2 = x_1 + x_3 + x_4 \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_1 + x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_1, x_3, x_4 \in \mathbb{R} \right\} \\ &= \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} : x_1, x_3, x_4 \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

   It is easy to check that these three vectors are linearly independent, so

$$\{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   is a basis for E. We now apply the Gram-Schmidt process to find an orthogonal (not necessarily orthonormal) basis $\{v_1, v_2, v_3\}$ for E:

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 - \operatorname{proj}_{v_1}(w_2) \\ v_3 &= w_3 - \operatorname{proj}_{v_1}(w_3) - \operatorname{proj}_{v_2}(w_3). \end{aligned}$$

   To make our calculations a bit easier, we will scale v_1, v_2, and v_3 so that they do not contain fractions. Notice that scaling the vectors does not change whether they are orthogonal.

Calculate v_2:

$$\begin{aligned} v_2 &= w_2 - \text{proj}_{v_1}(w_2) \\ &= w_2 - \frac{(w_2, v_1)}{(v_1, v_1)} v_1 \\ &= w_2 - \frac{1}{2} v_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Since v_2 contains fractions, we will scale it by a factor of 2. Multiply v_2 by 2. The new v_2 is given by

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Now calculate v_3:

$$\begin{aligned} v_3 &= w_3 - \text{proj}_{v_1}(w_3) - \text{proj}_{v_2}(w_3) \\ &= w_3 - \frac{(w_3, v_1)}{(v_1, v_1)} v_1 - \frac{(w_3, v_2)}{(v_2, v_2)} v_2 \\ &= w_3 - \frac{1}{2} v_1 - \frac{1}{2} v_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \end{aligned}$$

Multiplying by 3 gives the new v_3:

$$v_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

Therefore

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

is an orthogonal basis for E.

To find an orthonormal basis $\{e_1, e_2, e_3\}$ for E, we need to divide each of the vectors in our orthogonal basis by its norm:

$$e_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{(v_1, v_1)}}v_1 = \frac{1}{\sqrt{2}}v_1 = \frac{\sqrt{2}}{2}v_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$e_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{\sqrt{(v_2, v_2)}}v_2 = \frac{1}{\sqrt{6}}v_2 = \frac{\sqrt{6}}{6}v_2 = \frac{\sqrt{6}}{6}\begin{bmatrix}-1\\1\\2\\0\end{bmatrix}$$

$$e_3 = \frac{1}{\|v_3\|}v_3 = \frac{1}{\sqrt{(v_3, v_3)}}v_3 = \frac{1}{\sqrt{12}}v_3 = \frac{\sqrt{3}}{6}v_3 = \frac{\sqrt{3}}{6}\begin{bmatrix}-1\\1\\-1\\3\end{bmatrix}.$$

Therefore

$$\{e_1, e_2, e_3\} = \left\{\frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \frac{\sqrt{6}}{6}\begin{bmatrix}-1\\1\\2\\0\end{bmatrix}, \frac{\sqrt{3}}{6}\begin{bmatrix}-1\\1\\-1\\3\end{bmatrix}\right\}$$

is an orthonormal basis for E.

(b) Find the projection of vector $u = \begin{bmatrix}1\\0\\-1\\2\end{bmatrix}$ on E.

Solution: We will use the orthogonal basis

$$\{v_1, v_2, v_3\} = \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\3\end{bmatrix}\right\}$$

for E that we found in part (a). The projection of u onto E is

$$\begin{aligned}
\text{proj}_E(u) &= \text{proj}_{v_1}(u) + \text{proj}_{v_2}(u) + \text{proj}_{v_3}(u)\\
&= \frac{(u, v_1)}{v_1, v_1}v_1 + \frac{(u, v_2)}{v_2, v_2}v_2 + \frac{(u, v_3)}{v_3, v_3}v_3\\
&= \frac{1}{2}v_1 + \frac{-3}{6}v_2 + \frac{6}{12}v_3\\
&= \frac{1}{2}(v_1 - v_2 + v_3)\\
&= \frac{1}{2}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \begin{bmatrix}-1\\1\\2\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\-1\\3\end{bmatrix}\right)\\
&= \frac{1}{2}\begin{bmatrix}1\\1\\-3\\3\end{bmatrix}.
\end{aligned}$$

2. Let $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider the vector space $E = \{A \in M_{2\times 2}(\mathbb{R}) : AC = CA\}$.

   Find the projection of the vector $u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ onto E.

   Solution: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   then

$$AC = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \qquad \text{and} \qquad CA = \begin{bmatrix} c & d \\ a & b \end{bmatrix},$$

   so $AC = CA$ implies $b = c$ and $a = d$. We can then write E as

$$\begin{aligned} E &= \{A \in M_{2\times 2}(\mathbb{R}) : AC = CA\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}. \end{aligned}$$

   Since neither of these vectors is a scalar multiple of the other, this set is linearly independent, so

$$\{V_1, V_2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

   is a basis for E. The next step would be to find an orthogonal basis for E using the Gram-Schmidt process. However, notice that V_1 and V_2 are already orthogonal:

$$(V_1, V_2) = (1)(0) + (0)(1) + (0)(1) + (1)(0) = 0.$$

   Therefore $\{V_1, V_2\}$ is an orthogonal basis for E. We can now calculate the projection of u onto E:

$$\begin{aligned} \text{proj}_E(u) &= \text{proj}_{V_1}(u) + \text{proj}_{V_2}(u) \\ &= \frac{(u, V_1)}{(V_1, V_1)} V_1 + \frac{(u, V_2)}{(V_2, V_2)} V_2 \\ &= \frac{5}{2} V_1 + \frac{5}{2} V_2 \\ &= \frac{5}{2}(V_1 + V_2) \\ &= \frac{5}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{5}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

3. Let $V = \{u\colon [0,1] \to \mathbb{R},\ \ u \text{ is continuous}\}$ equipped with the inner product given by

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

Find the polynomial in $P_3(\mathbb{R})$ that is closest to vector $f(x) = e^x$. In other words, find a polynomial P of degree ≤ 3 such that $\|P - f\|$ is minimized.

Solution: There are many inner product calculations that need to be performed for this problem. We will omit these calculations.

We want to find the projection of the vector f onto $P_3(\mathbb{R})$. To do this, we first need an orthogonal basis for $P_3(\mathbb{R})$. Start with the ordinary basis

$$\{w_1, w_2, w_3, w_4\} = \{1, x, x^2, x^3\}$$

for $P_3(\mathbb{R})$. Now apply the Gram-schmidt process to find an orthogonal basis. Let $v_1 = w_1 = 1$. We now calculate v_2:

$$\begin{aligned} v_2 &= w_2 - \text{proj}_{v_1}(w_1) \\ &= x - \frac{(x,1)}{(1,1)}1 \\ &= x - \frac{1}{2}. \end{aligned}$$

Future calculations will be easier without fractions, so we will redefine v_2 by multiply by 2:

$$v_2 = 2x - 1.$$

Continuing with the Gram-Schmidt process and rescaling to get rid of fractions gives the basis $\{v_1, v_2, v_3, v_4\}$ for $P_3(\mathbb{R})$, where

$$\begin{aligned} v_1 &= 1 \\ v_2 &= 2x - 1 \\ v_3 &= 6x^2 - 6x + 1 \\ v_4 &= 20x^3 - 30x^2 + 12x - 1. \end{aligned}$$

Now we can use this orthogonal basis to find $\text{proj}_{P_3(\mathbb{R})}(g)$:

$$\begin{aligned} \text{proj}_{P_3(\mathbb{R})}(g) &= \text{proj}_{v_1}(g) + \text{proj}_{v_2}(g) + \text{proj}_{v_3}(g) + \text{proj}_{v_4}(g) \\ &= \frac{(g,v_1)}{(v_1,v_1)}v_1 + \frac{(g,v_2)}{(v_2,v_2)}v_2 + \frac{(g,v_3)}{(v_3,v_3)}v_3 + \frac{(g,v_4)}{(v_4,v_4)}v_4 \\ &= \frac{e-1}{1}(1) + \frac{3-e}{\frac{1}{3}}(2x-1) + \frac{7e-19}{\frac{1}{5}}(6x^2-6x+1) \\ &\quad + \frac{193-71e}{\frac{1}{7}}(20x^3 - 30x^2 + 12x - 1) \\ &= (27020 - 9940e)x^3 + (-41100 + 15120e)x^2 + (16800 - 6180e)x + (-1456 + 536e) \\ &\approx 0.2786x^3 + 0.4212x^2 + 1.0183x + 0.9991. \end{aligned}$$

4. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}.$

(a) Show that the equation $Ax = b$ has no solutions $x \in \mathbb{R}^2$.

Solution: Applying row reduction to the augmented system $Ax = b$ gives

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

The third line from the reduced matrix gives

$$0x_1 + 0x_2 = 1,$$

which is impossible. Therefore $Ax = b$ has no solutions.

(b) Although the equation has no solutions, one can attempt to find a least squares solution, $x \in \mathbb{R}^2$ that minimizes $\|Ax - b\|$.

(b.1) Find the projection of b on the column space col(A) (which is the range of A if viewing A as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^4$). Call it $\tilde{b}$.

Solution: Since neither column of A is a scalar multiple of the other, the columns are linearly independent, so they form a basis $\{w_1, w_2\}$ for col(A):

$$\{w_1, w_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}. \right\}$$

We can now use the Gram-Schmidt process to find an orthogonal basis $\{v_1, v_2\}$ for col(A). Let $v_1 = w_1$. Then

$$\begin{aligned} v_2 &= w_2 - \text{proj}_{v_1}(w_2) \\ &= w - 2 - \frac{(w_2, v_1)}{(v_1, v_1)} v_1 \\ &= w_2 - \frac{2}{9} v_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 18 \\ 9 \\ 9 \\ -9 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix}. \end{aligned}$$

To make future calculations easier, rescale v_2 by a factor of 9:

$$v_2 = \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix}.$$

Our orthogonal basis for col(A) is

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix} \right\}.$$

We can now calculate the projection of b onto col(A):

$$\begin{aligned} \tilde{b} &= \text{proj}_{\text{col}(A)}(b) \\ &= \text{proj}_{v_1}(b) + \text{proj}_{v_2}(b) \\ &= \frac{(b, v_1)}{(v_1, v_1)} v_1 + \frac{(b, v_2)}{(v_2, v_2)} v_2 \\ &= \frac{7}{9} v_1 + \frac{-32}{531} v_2 \\ &= \frac{7}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} - \frac{32}{531} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix} \\ &= \frac{1}{59} \begin{bmatrix} -11 \\ -32 \\ 74 \\ 138 \end{bmatrix}. \end{aligned}$$

(b.2) Solve for the exact solution of $Ax = \tilde{b}$.
Solution: We can solve this equation by applying row reduction to the equivalent augmented system:

$$\left[\begin{array}{cc|c} 1 & 2 & -\frac{11}{59} \\ 0 & 1 & -\frac{32}{59} \\ 2 & 1 & \frac{74}{59} \\ 2 & -1 & \frac{138}{59} \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{53}{59} \\ 0 & 1 & -\frac{32}{59} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore

$$x = \begin{bmatrix} \frac{53}{59} \\ -\frac{32}{59} \end{bmatrix}.$$

5. Consider $V = M_{3\times 3}(\mathbb{R})$. Find an orthogonal basis for the subspace

$$U = \text{span}\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Solution: First, let

$$\{W_1, W_2, W_3\} = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

We will use the Gram-Schmidt process to find an orthogonal basis $\{V_1, V_2, V_3\}$:

$$\begin{aligned} V_1 &= W_1 \\ V_2 &= W_2 - \text{proj}_{V_1}(W_2) \\ V_3 &= W_3 - \text{proj}_{V_1}(W_3) - \text{proj}_{V_2}(W_3). \end{aligned}$$

To calculate V_2 we will first need to calculate two inner products:

$$\begin{aligned} (W_2, V_1) = &\quad (1)(1) + (1)(0) + (0)(1) \\ &+ (0)(0) + (1)(1) + (0)(0) \\ &+ (0)(0) + (0)(0) + (1)(1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} (W_2, V_1) = &\quad (1)(1) + (1)(1) + (0)(0) \\ &+ (0)(0) + (1)(1) + (0)(0) \\ &+ (0)(0) + (0)(0) + (1)(1) \\ &= 4 \end{aligned}$$

We can now calculate V_2:

$$\begin{aligned} V_2 &= W_2 - \text{proj}_{V_1}(W_2) \\ &= W_2 - \frac{(W_2, V_1)}{(V_1, V_1)} V_1 \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

To make future calculations easier, we will rescale V_2 by multiplying by 4. The new V_2 is

$$V_2 = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now calculate V_3:

$$\begin{aligned} V_3 &= W_3 - \text{proj}_{V_1}(W_3) - \text{proj}_{V_2}(W_3) \\ &= W_3 - \frac{(W_3, V_1)}{(V_1, V_1)} V_1 - \frac{(W_3, V_2)}{(V_2, V_2)} V_2 \\ &= W_3 - \frac{3}{4} V_1 - \frac{3}{28} V_2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{28} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Although we have no more calculations to perform, our answer might look a bit nicer if we rescale V_3 as well. Multiply V_3 by 7 to get the new V_3:

$$V_3 = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore an orthogonal basis for U is

$$\{V_1, V_2, V_3\} = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$