Homework 7

Due 03/06/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

We will be using the natural inner product on $\mathbb{R}^n$ and $M_{m\times n}(\mathbb{R})$. That is,

$$\begin{aligned}(x,y) &= x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{j=1}^{n} x_jy_j, \\ (A,B) &= a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{mn}b_{mn} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}b_{jk}.\end{aligned}$$

1. Consider the vector space

$$E = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \quad x_2 = x_1 + x_3 + x_4 \right\}.$$

   (a) Find the projection of vector $u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ on E.

   (b) Find an orthonormal basis of E.

2. Let $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider the vector space

$$E = \{A \in M_{2\times 2}(\mathbb{R}) : \quad AC = CA\}.$$

   Find the projection of vector $u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ on E.

3. Let $V = \{u : [0,1] \to \mathbb{R}, \quad u \text{ is continuous}\}$. We equip V with an inner product defined by

$$(u,v) = \int_0^1 u(x)v(x)dx.$$

   Find the polynomial in $P_3(\mathbb{R})$ that is closest to vector $f(x) = e^x$. In other words, find a polynomial P of degree ≤ 3 such that $\|P - f\|$ is minimum.

4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

   (a) Show that the equation $Ax = b$ has no solutions $x \in \mathbb{R}^2$.

   (b) Although the equation has no solutions, one can attempt to find the "best" approximate solution. The best approximate solution is defined as $x \in \mathbb{R}^2$ that minimizes $\|Ax - b\|$. It can be found through two following steps.

(b.1) Find the projection of b on the column space $\text{col}(A)$ (which is the range of A if viewing A as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^4$). Call it $\tilde{b}$.

(b.2) Solve for the exact solution of $Ax = \tilde{b}$.

Do the following problem for 6 bonus points. You are encouraged to use computer software (Matlab, Mathematica, etc.) – Turn in your code with the solution.

5. Consider $V = M_{3\times 3}(\mathbb{R})$. Find an orthogonal basis of the subspace

$$U = \text{span}\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Hint: Use Gram-Schmidt orthogonalization procedure. Start with the first matrix as part of your basis. Calculate the orthogonal projection of the second matrix onto the first matrix and subtract that from the second matrix. This results in a second matrix for your basis. Calculate the orthogonal projections of the third matrix onto both matrices you have in your basis, and subtract both from the third matrix. This matrix completes your basis.