Homework 8
Answer Key

1. Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$ where

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$.

   Solution: To find a least squares solution, we need to solve the equation $A^*A\mathbf{x} = A^*\mathbf{b}$. In this case,

$$A^* = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix},$$

   so

$$A^*A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} \qquad \text{and} \qquad A^*\mathbf{b} = \begin{bmatrix} 58 \\ 74 \end{bmatrix}.$$

   We can solve for $\mathbf{x}$ by applying row recuction to the augmented system:

$$\left[\begin{array}{cc|c} 35 & 44 & 58 \\ 44 & 56 & 74 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{19}{12} \end{array}\right].$$

   Therefore

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{3} \\ \frac{19}{12} \end{bmatrix}$$

   is the least squares solution to the system.

   (b) $A = \begin{bmatrix} i & 1 \\ 2 & -2i \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} i+1 \\ 1 \end{bmatrix}$.

   Solution: We again need to solve the equation $A^*A\mathbf{x} = A^*\mathbf{b}$. In this case,

$$A^* = \begin{bmatrix} -i & 2 \\ 1 & 2i \end{bmatrix},$$

   so

$$A^*A = \begin{bmatrix} 5 & -5i \\ 5i & 5 \end{bmatrix} \qquad \text{and} \qquad A^*\mathbf{b} = \begin{bmatrix} 3-i \\ 1+3i \end{bmatrix}.$$

   We can solve for $\mathbf{x}$ by applying row recuction to the augmented system:

$$\left[\begin{array}{cc|c} 5 & -5i & 3-i \\ 5i & 5 & 1+3i \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -i & \frac{3}{5}-\frac{1}{5}i \\ 0 & 0 & 0 \end{array}\right].$$

   Therefore

$$x_1 - ix_2 = \frac{3}{5} - \frac{1}{5}i.$$

   Solving for x_1 gives

$$x_1 = \frac{3}{5} - \frac{1}{5}i + ix_2.$$

   There are thus infinitely many least squares solutions. They can be written as

$$\mathbf{x} = \begin{bmatrix} \frac{3}{5} - \frac{1}{5}i + ix_2 \\ x_2 \end{bmatrix}$$

   where $x_2 \in \mathbb{C}$.

2. Find a plane $z = ax + by + c$ that best fits the five points $(1,1,1)$, $(1,2,4)$, $(2,0,1)$, $(-1,2,7)$ and $(5,4,1)$. Is the answer unique?

Solution: We first need to set up a system $A\vec{x} = \vec{b}$ where We want to find a least squares solution

$$\begin{aligned} x_1 a + y_1 b + c &= z_1 \\ &\vdots \\ x_n a + y_n b + c &= z_n \end{aligned}$$

Notice that this system can be written as a matrix equation $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} x_1 & y_1 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

In our case we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & 1 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 7 \\ 1 \end{bmatrix}.$$

To find a least squares solution, we need to solve the system $A^*A\vec{x} = A^*\vec{b}$. First we find A^*:

$$A^* = \begin{bmatrix} 1 & 1 & 2 & -1 & 5 \\ 1 & 2 & 0 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now

$$A^*A = \begin{bmatrix} 32 & 21 & 8 \\ 21 & 25 & 9 \\ 8 & 9 & 5 \end{bmatrix} \qquad \text{and} \qquad A^*\vec{b} = \begin{bmatrix} 19 \\ -1 \\ 0 \end{bmatrix}$$

We can now solve the system using row reduction:

$$\left[\begin{array}{ccc|c} 32 & 21 & 8 & 19 \\ 21 & 25 & 9 & -1 \\ 8 & 9 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{79}{57} \\ 0 & 1 & 0 & -\frac{241}{209} \\ 0 & 0 & 1 & -\frac{89}{627} \end{array}\right].$$

This system therefore has a unique solution:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{79}{57} \\ -\frac{241}{209} \\ -\frac{89}{627} \end{bmatrix}.$$

The equation for the plane of best fit is

$$z = \frac{79}{57}x - \frac{241}{209}y - \frac{89}{627}.$$

3. Let $A, B \in M_{n\times n}(\mathbb{C})$. Recall that A is *unitary* if $A^{-1} = A^*$. Show that if A, B are unitary then AB is unitary.

Solution: Recall that for any matrices $A, B \in M_{n\times n}(\mathbb{C})$ we have

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{and} \qquad (AB)^* = B^*A^*.$$

Now if A, B are unitary, then

$$\begin{aligned}(AB)^* &= B^*A^* \\ &= B^{-1}A^{-1} \qquad \text{since } A, B \text{ are unitary} \\ &= (AB)^{-1}.\end{aligned}$$

Therefore AB is unitary.

4. Let $A \in M_{m\times n}(\mathbb{C})$. Show that

   (a) The eigenvalues of A^*A are real and non-negative.

   Solution: Let $(\cdot,\cdot)_n$ denote the standard inner product on $\mathbb{C}^n$. Recall that for any matrix $A \in\in M_{m\times n}$, the adjoint A^* satisfies

$$(A^*x, y)_m = (x, Ay)_n$$

   Let v be an eigenvector of A^*A with associated eigenvalue λ. We will show separately that λ is real and that λ is non-negative.

   (i) **$\boldsymbol{\lambda}$ is real:** Let $B = A^*A$. notice that

$$B^* = (A^*A)^* = (A)^*(A^*)^* = A^*A = B,$$

   so B is self-adjoint. Then

$$\begin{gathered}(B^*v, v) = (v, Bv) \\ \downarrow \\ (Bv, v) = (v, Bv) \\ \downarrow \\ (\lambda v, v) = (v, \lambda v) \\ \downarrow \\ \lambda(v, v) = \overline{\lambda}(v, v) \\ \downarrow \\ \lambda = \overline{\lambda}.\end{gathered}$$

   Notice that in the last step we divided by (v, v). We can do this, becuase know that (v, v) is nonzero, since v is an eigenvector (and hence v is nonzero). Since λ is equal to its own complex conjugate, it must be real.

(ii) **$\boldsymbol{\lambda}$ is non-negative:**

$$(A^*Av, v) = (Av, (A^*)^*v)$$
$$\downarrow$$
$$(A^*Av, v) = (Av, Av)$$
$$\downarrow$$
$$(\lambda v, v) = (Av, Av)$$
$$\downarrow$$
$$\lambda(v, v) = (Av, Av)$$
$$\downarrow$$
$$\lambda = \frac{(Av, Av)}{(v, v)}.$$

Since $(Av, Av) \geq 0$ and $(v, v) > 0$ by positivity and definiteness of the inner product, we get

$$\lambda = \frac{(Av, Av)}{(v, v)} \geq 0,$$

so λ is non-negative.

(b) $\text{null}(A^*A + I_n) = \{0\}$.

Solution: Recall that the eigenvalues λ of $B \in M_{n\times n}(\mathbb{C})$ are the only values such that

$$\text{null}(B - \lambda I) \neq \{0\}.$$

Since -1 is not an eigenvalue of A^*A. we must have

$$\text{null}(A^*A + I_n) = \text{null}(A^*A - (-1)I_n) = \{0\}.$$

(c) The matrix $A^*A + I_n$ is invertible.

Solution: Since $\text{null}(A^*A + I_n) = \{0\}$ the map $A^*A \colon \mathbb{C}^n \to \mathbb{C}^n$ is injective, and hence invertible. Therefore the matrix A^*A is invertible.

5. Consider the real inner product space $V = \mathbb{P}_2(\mathbb{R})$ with inner product given by

$$(f(x), g(x)) = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Find an orthonormal basis of V.

Solution: Apply the Gram-Schmidt to the basis $\{1, x, x^2\}$ to find an orthogonal basis $\{w_1, w_2.w_3\}$:

$$w_1 = 1$$

$$\begin{aligned} w_2 &= x - \mathrm{proj}_{w_1}(x) \\ &= x - \frac{(x,1)}{(1,1)} \cdot 1 \\ &= x - \frac{0}{2} \cdot 1 \\ &= x. \end{aligned}$$

$$\begin{aligned} w_3 &= x^2 - \mathrm{proj}_{w_1}(x^2) - \mathrm{proj}_{w_2}(x^2) \\ &= x^2 - \frac{(x^2,1)}{(1,1)} \cdot 1 - \frac{(x^2,x)}{(x,x)} \cdot x \\ &= x^2 - \frac{\frac{2}{3}}{2} \cdot 1 - \frac{0}{1} \cdot x \\ &= x^2 - \frac{2/3}{2} \cdot 1 - \frac{0}{1} \cdot x \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

To make future calculations easier, we will rescale w_3 by a factor of 3 (i.e., multiply it by 3). The new w_3 is then

$$w_3 = 3x^2 - 1$$

We can now find an orthonormal basis $\{v_1, v_2, v_3\}$ by normalizing each vector in the orthogonal basis $\{w_1, w_2, w_3\}$:

$$v_1 = \frac{1}{\|w_1\|} w_1 = \frac{1}{\sqrt{(1,1)}} \cdot (1) = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

$$v_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{(x,x)}} \cdot (x) = \frac{1}{1}(x) = x$$

$$v_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{\sqrt{(3x^2-1, 3x^2-1)}} \cdot 1 = \frac{1}{\sqrt{8/5}} \cdot (3x^2 - 1) = \frac{\sqrt{10}}{4}(3x^2 - 1)$$

Thus an orthonormal basis for $P_2(\mathbb{R})$ is

$$\{v_1, v_2, v_3\} = \left\{ \frac{1}{\sqrt{2}},\ x,\ \frac{\sqrt{10}}{4}(3x^2 - 1) \right\}.$$

(b) Consider the derivative operator $D : V \to V$ given by $D(u) = u'$. Determine the adjoint operator D^*.

Solution: We first want to find the matrix representation for D with respect to the orthonormal basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \frac{1}{\sqrt{2}},\ x,\ \frac{\sqrt{10}}{4}(3x^2 - 1) \right\}$$

that we found part (a). Applying D to each element of the basis gives

$$D(v_1) = 0$$

$$D(v_2) = 1 = \sqrt{2} v_1$$

$$D(v_3) = \frac{3\sqrt{10}}{2} x = \frac{3\sqrt{10}}{2} v_2.$$

Therefore the matrix representation of D with respect to the basis $\mathcal{B}$ is

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \frac{3\sqrt{10}}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The adjoint matrix is

$$([D]_{\mathcal{B}})^* = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \frac{3\sqrt{10}}{2} & 0 \end{bmatrix}.$$

We can use this matrix to find D^* applied to the standard basis $\{1, x, x^2\}$. First, write these elements in $\mathcal{B}$ coordinates:

$$1 = \sqrt{2} v_1 \quad \to \quad [1]_{\mathcal{B}} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix},$$

$$x = v_2 \quad \to \quad [x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$x^2 = \tfrac{2\sqrt{10}}{15} v_3 + \tfrac{\sqrt{2}}{3} v_1 \quad \to \quad [x^2]_{\mathcal{B}} = \begin{bmatrix} \frac{\sqrt{2}}{3} \\ 0 \\ \frac{2\sqrt{10}}{15} \end{bmatrix}.$$

We can now calculate D^* applied to the standard basis vectors written in $\mathcal{B}$ coordinates.

$$[D^*(1)]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \frac{3\sqrt{10}}{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

$$[D^*(x)]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \frac{3\sqrt{10}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{3\sqrt{10}}{2} \end{bmatrix},$$

$$[D^*(x^2)]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \frac{3\sqrt{10}}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} \\ 0 \\ \frac{2\sqrt{10}}{15} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \end{bmatrix}.$$

Converting back to polynomials gives

$$\begin{aligned} D^*(1) &= 2v_1 \\ &= 2x, \end{aligned}$$

$$\begin{aligned} D^*(x) &= \frac{3\sqrt{10}}{2} v_3 \\ &= \frac{3\sqrt{10}}{2} \cdot \frac{\sqrt{10}}{4}(3x^2 - 1) \\ &= \frac{15}{4}(3x^2 - 1) \end{aligned}$$

$$\begin{aligned} D^*(x^2) &= \frac{2}{3} v_1 \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{3}. \end{aligned}$$

Since D^* is a linear operator, we can calculate D^* for any polynomial $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$:

$$\begin{aligned} D^*(ax^2 + bx + c) &= aD^*(x^2) + bD^*(x) + cD^*(1) \\ &= a\left(\frac{\sqrt{2}}{3}\right) + b\left(\frac{15}{4}(3x^2 - 1)\right) + c(2x). \end{aligned}$$

6. Find a singular value decomposition of each square matrix below.

(a) $A = \begin{bmatrix} 3 & -2 \\ -6 & -1 \end{bmatrix}$

Solution: The goal is to find unitary matrices P, Q and the diagonal matrix D of singular values such that

$$A = PDQ^*.$$

To find the diagonal matrix D, we need to find the singular values of A. First, find the eigenvalues of

$$A^*A = \begin{bmatrix} 45 & 0 \\ 0 & 5 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 45$ and $\lambda_2 = 5$. The singular values σ_1, σ_2 of A are then the square roots of the eigenvalues of A^*A:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{45} = 3\sqrt{5} \qquad \text{and} \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt{5}$$

The matrix D is the diagonal matrix with the singular values along the diagonal:

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}.$$

Next we want to find the matrix Q. The columns of Q are an orthonormal basis of eigenvectors for A^*A. Recall that the eigenvalues of A^*A are $\lambda_1 = 45$ and $\lambda_2 = 5$. The eigenspace associated to each eigenvalue has basis element

$$\lambda_1 = 45\colon \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 5\colon \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that both of these vectors have magnitude 1. If they did not, we would have to normalize them (i.e., divide them by their magnitude). Therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is the desired orthonormal basis for A^*A. The matrix Q is then

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice that the order of the columns in Q matches the order of the singular values in D.

Finally, we want to find the unitary matrix P. We can find the kth column p_k of P using the formula

$$p_k = \frac{1}{\sigma_k} A q_k$$

where σ_k is the kth singular value and q_k is the kth column of Q. We get

$$\begin{aligned} p_1 &= \frac{1}{3\sqrt{5}} \begin{bmatrix} 3 & -2 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{3\sqrt{5}} \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \end{aligned} \qquad \begin{aligned} p_2 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & -2 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}. \end{aligned}$$

Therefore

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

A singular value decomposition of A is then

$$A = PDQ^* = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Solution: We want to find unitary matrices P, Q and the diagonal matrix D of singular values such that

$$B = PDQ^*.$$

We first find D. Calculate B^*B:

$$B^*B = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}.$$

The eigenvalues of B^*B are $\lambda_1 = 10$ and $\lambda_2 = 0$. Therefore the singular values are $\sigma_1 = \sqrt{10}$ and $\sigma_2 = 0$. The diagonal matrix D is given by

$$D = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}.$$

Next we want to find Q. The eigenspace associated to each eigenvalue of B^*B has basis element

$$\lambda_1 = 10\colon \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \lambda_2 = 0\colon \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

We want unit eigenvectors, so we need to divide each vector by its magnitude:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Therefore

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \qquad \text{and} \qquad Q^* = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Finally, we need to find P. We can find the first column p_1 of P using the formula

$$p_k = \frac{1}{\sigma_k} B v_k$$

where σ_k is the kth singular value and p_k and q_k are the kth columns of P and Q respectively. The calculation for p_1 is

$$\begin{aligned} p_1 &= \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}. \\ &= \frac{1}{\sqrt{50}} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{5\sqrt{2}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Notice that we cannot use the formula to find p_2, because $\sigma_2 = 0$. Instead, we need to find a unit vector that is orthogonal to p_1. The standard way to do this is to pick some vector x such that x and p_1 are linearly independent. We would then calculate the vector

$$y = x - \text{proj}_{p_1}(x).$$

The second row p_2 of P is then

$$p_2 = \frac{1}{\|y\|} y.$$

I will skip most of this process by noting that the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is orthogonal to p_1. Normalizing this vector (i.e., dividing it by its norm) gives

$$p_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix P is

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

A singular value decomposition of B is then

$$B = PDQ^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

(c) $C = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$.

Solution: We want to find unitary matrices P, Q and the diagonal matrix D of singular values such that

$$C = PDQ^*.$$

We first find D. Calculate C^*C:

$$C^*C = \begin{bmatrix} 6 & 3-3i \\ 3+3i & 3 \end{bmatrix}.$$

The eigenvalues of C^*C are $\lambda_1 = 9$ and $\lambda_2 = 0$. Therefore the singular values are $\sigma_1 = 3$ and $\sigma_2 = 0$. The diagonal matrix D is given by

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

Next we want to find Q. The eigenspace associated to each eigenvalue of C^*C has basis element

$$\lambda_1 = 9\colon \begin{bmatrix} 1-i \\ 1 \end{bmatrix}, \qquad \lambda_2 = 0\colon \begin{bmatrix} 1 \\ -1-i \end{bmatrix}.$$

We want unit eigenvectors, so we need to divide each vector by its magnitude. Note that

$$\left\| \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\| = \sqrt{(1-i)\overline{(1-i)} + (1)\overline{(1)}} = \sqrt{(1-i)(1+i) + (1)(1)} = \sqrt{2+1} = \sqrt{3}$$

and

$$\left\| \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \right\| = \sqrt{(1)\overline{(1)} + (-1-i)\overline{(-1-i)}} = \sqrt{(1)(1) + (-1-i)(-1+i)} = \sqrt{1+2} = \sqrt{3},$$

so

$$\begin{bmatrix} 1-i \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \to \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{-1-i}{\sqrt{3}} \end{bmatrix}.$$

Therefore

$$Q = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1-i}{\sqrt{3}} \end{bmatrix} \quad \text{and} \quad Q^* = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{-1+i}{\sqrt{3}} \end{bmatrix}.$$

Finally, we need to find P. We can find the first column p_1 of P using the formula

$$p_k = \frac{1}{\sigma_k} C v_k$$

where σ_k is the kth singular value and p_k and q_k are the kth columns of P and Q respectively. The calculation for p_1 is

$$\begin{aligned} p_1 &= \frac{1}{3} \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}. \\ &= \frac{1}{3\sqrt{3}} \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \\ &= \frac{1}{3\sqrt{3}} \begin{bmatrix} 3-3i \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}. \end{aligned}$$

Notice that $p_1 = q_1$. We want to choose p_2 such that p_1 and p_2 are orthogonal, so just choose $p_2 = q_2$ (this works since q_2 is orthogonal to $q_1 = p_1$). That is,

$$p_2 = q_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{-1-i}{\sqrt{3}} \end{bmatrix}$$

Therefore

$$P = Q = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1-i}{\sqrt{3}} \end{bmatrix}.$$

A singular value decomposition of C is then

$$C = PDQ^* = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1-i}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{-1+i}{\sqrt{3}} \end{bmatrix}.$$