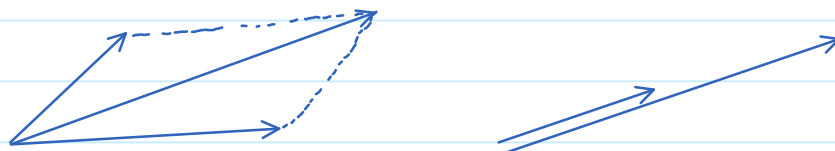


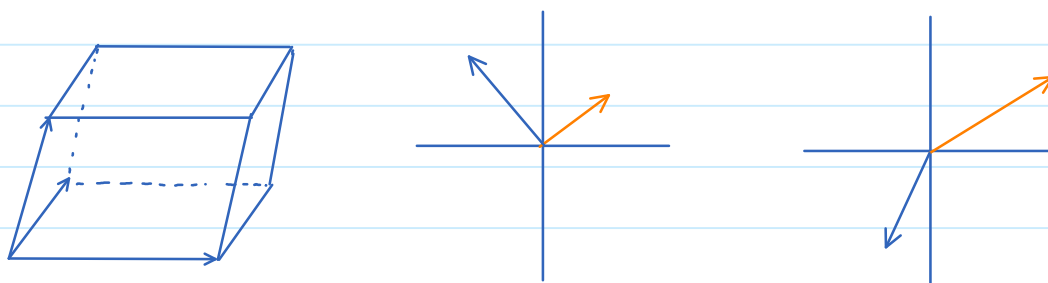
# Lecture 1

Monday, January 6, 2020

In Linear Algebra I, we learned vector spaces from *descriptive* perspective. In other words, we defined a vector space by pointing what it is. The vector spaces we studied were  $\mathbb{R}^n$  and  $\mathbb{C}^n$ . This approach is quite intuitive. For example, one can visualize a vector as an arrow.



The addition and scaling of vectors can be visualized. The determinant can also be visualized as volume of a parallelepiped. The eigenvectors are directions that are invariant under a transformation.



In this course, we will learn Linear Algebra from an *axiomatic* approach. Instead of working on an exclusive vector space  $\mathbb{R}^n$ , we will work with more general vector spaces. A vector space is a set that satisfies certain properties. These properties can be split into 3 groups:

- (A): addition properties
  - (S): scaling properties
  - (D): distribution properties
- } will be discussed in detail next time

The advantage of the axiomatic approach is that it makes the linear algebra tools we defined on  $\mathbb{R}^n$  more useful. For example,

one can talk about addition, scaling, linear combination, linear independence, determinant, eigenvectors, ... for a more variety of sets, such as

$M_{m \times n}$  : the set of matrices of size  $m \times n$ ,

$\mathbb{R}^{[0,1]}$  : the set of functions from  $[0,1]$  to  $\mathbb{R}$ ,

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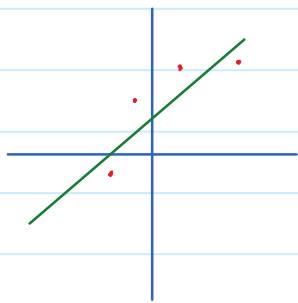
After the midterm example, we will cover new topics that were not included in Linear Algebra I. These include orthogonality and singular value decomposition. The idea of orthogonality is quite natural on  $\mathbb{R}^n$ : two vectors are perpendicular if their dot product is equal to zero.

For general vector spaces, this concept is not trivial to see. For example, what does it mean for two functions from  $[0,1]$  to  $\mathbb{R}$  to be perpendicular to each other?

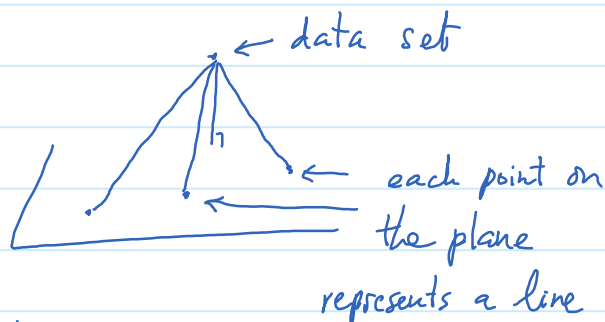
Axiomatic approach to linear algebra } First half  
Midterm

• Inner product space (orthogonality, ...) } second half  
• Singular value decomposition

The idea of orthogonality is very useful. An example is the least square method.



How to best fit a data set by a line?



The perpendicular projection of the data set on the "plane" is the solution.

Singular value decomposition is a useful tool in signal processing, especially in data compression. It extracts a small set of data from a large set of data but still preserves certain important features.

An important component of the class is writing proof. Consider an example.

Problem: Show that

$$x^2 + 1 \geq 2x \quad \forall x \in \mathbb{R}$$

Here is a possible solution:

$$x^2 + 1 - 2x \geq 0$$

$$(x-1)^2 \geq 0$$

This answer captures the main idea of the solution. However, it is not a proof for several reasons.

- 1) It starts with a formula instead of words
- 2) What is  $x$ ?
- 3) What is the transition between two lines?

A better solution is the following.

Let  $x \in \mathbb{R}$ . We want to show that  $x^2 + 1 \geq 2x$ .

This is equivalent to showing that  $x^2 + 1 - 2x \geq 0$ .

The left hand side can be factored as  $(x-1)^2$ , which is greater than or equal to 0.

A helpful tip is to look closely to the textbook: how did the author write his textbook? Note that he wrote in full sentences, beginning with an uppercase letter and ending with a period. We will have many opportunities to practice proof writing in this course.