Lecture 10

Wednesday, January 29, 2020

Let U and V be subspaces of F^k ($\mathbb{Q}^k$, $\mathbb{R}^k$ or $\mathbb{C}^k$). To find a basis of $U+V$, we do the following:

1) Find a basis of U, say $B_1 = \{u_1, u_2, \dots, u_n\}$
2) Find a basis of V, say $B_2 = \{v_1, v_2, \dots, v_m\}$.
3) Put $B = B_1 \sqcup B_2 = \{u_1, \dots, u_n, v_1, \dots, v_m\}$.

Then form matrix

$$A = \begin{bmatrix} | & | & & | & | & | & & | \\ u_1 & u_2 & \cdots & u_n & v_1 & v_2 & \cdots & v_m \\ | & | & & | & | & | & & | \end{bmatrix}$$

We have $U+V = \text{span}(B_1 \sqcup B_2) = \text{col}(A)$. We already know from Linear Algebra I how to find a basis of $\text{col}(A)$. Accordingly, we find the RREF of A.

$$A \longrightarrow \begin{bmatrix} & & & \\ & \cdots & & \\ & & & \end{bmatrix}$$

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pivot columns tell us which columns of A we should take to form a basis of $\text{col}(A)$.

In case U and V are abstract vector spaces, for example when their elements are matrices or polynomials, then we will need to transform the problem on the abstract space to a problem on F^k. This is done through coordinates. Consider the following example.

Ex:

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \ a = d = 0, \ b + c = 0 \right\}$$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \ a + d = 0, \ b + c = 0 \right\}$$

Find a basis of $U+V$.

There are two ways to do this problem.

- Method 1: use coordinates

We observe that U and V are subsets of $M_{2\times2}(\mathbb{R})$. We pick a basis of $M_{2\times2}(\mathbb{R})$ and convert the problem in terms of coordinates. Let us choose the standard basis of $M_{2\times2}(\mathbb{R})$:

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$$

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has cordinate $[A]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

because $A = aE_1 + bE_2 + cE_3 + dE_4$.

Subspace U of $M_{2\times2}(\mathbb{R})$ now corresponds to

$$U' = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \cdot \quad a = d = 0,\ b + c = 0 \right\}.$$

In other words, U' is the set of coordinates of vectors in U (with respect to basis $\mathcal{B}$). Similarly, subspace V of $M_{2\times2}(\mathbb{R})$ corresponds to

$$V' = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \cdot \quad a + d = 0,\ b + c = 0 \right\}.$$

Now the problem becomes: find a basis of $U' + V'$. Note that we are now working in $\mathbb{R}^4$, not in an abstract vector space anymore. We will follow the procedure set forth earlier.

- Find a basis of U'.

In the description of U', there are 3 constraints: $a = 0$, $d = 0$, $b + c = 0$

The idea is to try to remove all constraints, leaving only free variables. We can do so by substituting a and d by 0, and c by $-b$. Now b is a free variable

$$U' = \left\{ \begin{bmatrix} 0 \\ b \\ -b \\ 0 \end{bmatrix} : b \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} : b \in \mathbb{R} \right\} = \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{u_1'} \right\}$$

Thus, U' has basis $B_1' = \{u_1'\}$.

- Find basis of V'.

In the same spirit, we substitute $-a$ for d, and $-b$ for c, leaving a and b free variables.

$$V' = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{v_1'}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{v_2'} \right\}$$

To say that $\{v_1', v_2'\}$ is a basis of V', we need to check if they are linearly independent. Consider the equation $c_1 v_1' + c_2 v_2' = 0$ with unknowns c_1 and c_2. This equation is equivalent to

$$\begin{bmatrix} c_1 \\ c_2 \\ -c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies $c_1 = c_2 = 0$. Therefore, $\{v_1', v_2'\}$ is linearly independent. Thus, $B_2' = \{v_1', v_2'\}$ is a basis of V'.

- Find a basis of $U' + V'$.

We will remove "redundancy" from the set $B' = B_1' \sqcup B_2' = \{u_1', v_1', v_2'\}$

$$\begin{bmatrix} | & | & | \\ u_1' & v_1' & v_2' \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

One can do row reduction by hand or use the command rref in Matlab.

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>> A= [0 1 0; 1 0 1; -1 0 -1; 0 -1 0]
>> rref(A)
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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ $\uparrow$ pivot columns

Therefore, $\{u_1', v_1'\}$ is a basis of $U'+V'$.

We convert the result back to the abstract setting.

$$u_1' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ is the coordinate of matrix } A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \text{ is the coordinate of matrix } A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A basis of $U+V$ is $\{A_1, A_2\}$.

■ Method 2: free of coordinate

In this method, we only use the definition of $U+V$, without resorting to coordinates. We have

$$U+V = \{A+B \cdot A \in U, B \in V\}$$

$$= \left\{ A+B : A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \begin{array}{l} a_1 = d_1 = 0, b_1 + c_1 = 0 \\ a_2 + d_2 = b_2 + c_2 = 0 \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} : a_1 = d_1 = 0, b_1 + c_1 = 0, a_2 + d_2 = b_2 + c_2 = 0 \right\}$$

We try to eliminate all constraints, leaving only free variables This is done by substituting 0 for a_1 and d_1, $-b_1$ for c_1, $-b_2$ for c_2, $-a_2$ for d_2. Then we are left with only free variables a_2, b_1, b_2

$$U+V = \left\{ \begin{bmatrix} a_2 & b_1+b_2 \\ -b_1-b_2 & -a_2 \end{bmatrix} : a_2, b_1, b_2 \in \mathbb{R} \right\}$$

$$= \left\{ a_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : a_2, b_1, b_2 \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{repeated}} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

One can easily verify that these two abstract vectors are linearly independent
Therefore, they form a basis of $U+V$.