Lecture 11

Friday, January 31, 2020

Recall: Let U and V be vector spaces. Then $U+V$ is then smallest vector space that contains both U and V. How to find a basis of $U+V$?
We first find a basis of U, called $\mathcal{B}_1$, and a basis of V, called $\mathcal{B}_2$.

$$\mathcal{B}_1 = \{v_1, v_2, \ldots, v_n\},$$
$$\mathcal{B}_2 = \{w_1, w_2, \ldots, w_m\}.$$

Then we concatenate $\mathcal{B}_1$ and $\mathcal{B}_2$: $\mathcal{B}_1 \sqcup \mathcal{B}_2 = \{v_1, v_2, \ldots, v_n, w_1, \ldots, w_m\}$.
If these vectors are regular vectors (in F^k) then the next step is to align them as columns of a matrix.

$$\begin{bmatrix} | & | & & | & | & & | \\ v_1 & v_2 & \cdots & v_n & w_1 & \cdots & w_m \\ | & | & & | & | & & | \end{bmatrix}$$

and then do row reduction. The pivot columns indicate which vectors among $\mathcal{B}_1 \sqcup \mathcal{B}_2$ should be taken to form a basis of $U+V$.

If $v_1, v_2, \ldots, v_n, w_1, \ldots, w_m$ are abstract vectors (e.g. matrices, polynomials...) then we will choose a basis $\mathcal{C}$ for a vector space that contains both U and V. Then express $v_1, \ldots, v_n, w_1, \ldots, w_m$ in terms of coordinate relative to $\mathcal{C}$. Then do row reduction for the matrix:

$$\begin{bmatrix} \underbrace{\begin{matrix} | & | & & | \\ [v_1]_{\mathcal{C}} & [v_2]_{\mathcal{C}} & \cdots & [v_n]_{\mathcal{C}} \\ | & | & & | \end{matrix}}_{\substack{\text{basis vectors of} \\ U' \text{ (the "coordinate"} \\ \text{version of } U)}} & \underbrace{\begin{matrix} | & & | \\ [w_1]_{\mathcal{C}} & \cdots & [w_m]_{\mathcal{C}} \\ | & & | \end{matrix}}_{\substack{\text{basis vectors of } V' \\ \text{(the coordinate} \\ \text{version of } V)}} \end{bmatrix}$$

Once we get a basis for $U'+V'$, convert it to a basis of $U+V$.

* Direct sum:

The sum $U+V$ is called a direct sum if <u>one</u> of the following is satisfied:

1) $B_1 \sqcup B_2$ is linearly independent.

Here B_1 is a basis of U, and B_2 is a basis of V.
In other words, this is the situation when $B_1 \sqcup B_2$ has no "redundant" vectors. After doing RREF of the matrix, every column has to be a pivot column. The RREF looks like:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2) $\dim(U+V) = \dim U + \dim V$

This property is equivalent to 1). Suppose 1) is true. Then

$$\begin{aligned} \dim(U+V) &= \#(B_1 \sqcup B_2) \\ &= (\# B_1) + (\# B_2) \quad \text{(because no vectors in } B_1 \sqcup B_2 \text{ are thrown away)} \\ &= \dim U + \dim V. \end{aligned}$$

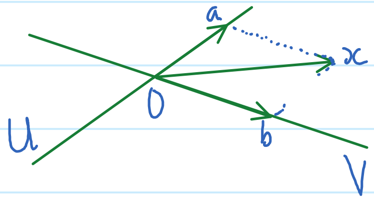
Thus, 2) is also true. This means that 1) infers 2) One can also show that 2) infers 1) without much difficulty.

3) $U \cap V = \{0\}$

This property says that: $U+V$ is called a direct sum is U and V has virtually nothing in common. We know that every vector space must contain a zero element. In case of direct sum, the zero element is the only element U and V have in common.

4) Each $x \in U+V$ can be written in a unique way as $x = u+v$ where $u \in U$ and $v \in V$.

This property comes from coordinate perspective. In this case, U and V can be regarded as "coordinate axes": each $x \in U+V$ has a unique U-coordinate and V-coordinate.

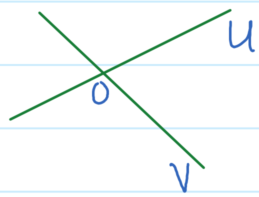


$x = a + b$

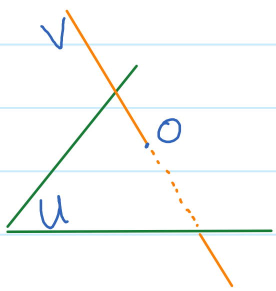
The statements 1), 2), 3), 4) are equivalent In other words, if one statement is true then the other three are also true. If one statement is false then the rest are also false

Notation: if $U+V$ is a direct sum then we write it as $U \oplus V$.

Ex:



$U+V$ is a direct sum because $U \cap V = \{0\}$.



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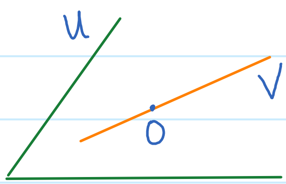

$U \cap V$ is not a direct sum because

$U \cap V = V \neq \{0\}$.

Ex: Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 = x_1 + x_3,\ x_3 = 2x_1 - x_2 + 5x_4\}$.

Find a subspace W of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$

Analyze the problem:

V is described as the set of vectors in $\mathbb{R}^4$ (which is 4-dimensional) that satisfy two conditions. Thus, it is likely that V is 2-dimensional In order to have a direct sum $V+W$, it is necessary that W is 2-dimesional. Why?

$$\dim W \underset{\substack{\uparrow \\ \text{due to} \\ \text{direct sum}}}{=} \dim(V+W) - \dim V = 4 - 2 = 2$$

To do this problem, we first find a basis of V. From the description of V, we see that it is the solution space of a system of 2 equations with 4 unknowns.

$$\begin{cases} x_1 - x_2 + x_3 = 0, \\ 2x_1 - x_2 - x_3 + 5x_4 = 0. \end{cases}$$

In matrix form:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 5 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 5 & 0 \\ 0 & 1 & -3 & 5 & 0 \end{array}\right]$$

The third and fourth columns give us free variables:

$$\begin{cases} x_3 = t, \\ x_4 = s, \\ x_2 = 3x_3 - 5x_4 = 3t - 5s, \\ x_1 = 2x_3 - 5x_4 = 2t - 5s. \end{cases}$$

Therefore, V is written as

$$\begin{aligned} V &= \{(2t-5s, 3t-5s, t, s) : t, s \in \mathbb{R}\} \\ &= \{ t(2,3,1,0) + s(-5,-5,0,1) : t, s \in \mathbb{R}\} \\ &= \operatorname{span}\{(2,3,1,0), (-5,-5,0,1)\}. \end{aligned}$$

We obtain a basis of V, namely $B_1 = \{\underbrace{(2,3,1,0)}_{v_1}, \underbrace{(-5,-5,0,1)}_{v_2}\}$.

Now we need to add two more vectors, say w_1 and w_2, to B_1 to make it a basis of $\mathbb{R}^4$. These vectors will form a basis for W.

$$B_2 = \{w_1, w_2\}.$$

Why so? Note that the concatenation $B_1 \sqcup B_2$ will be $\{v_1, v_2, w_1, w_2\}$, which is a linearly independent set. In particular, $V + W$ will be a direct sum. It will have dimension 4, and thus be equal to $\mathbb{R}^4$

How do we find w_1 and w_2? We arrange the basis vectors of V as <u>rows</u> of a matrix:

$$\begin{bmatrix} \text{---} \; v_1 \; \text{---} \\ \text{---} \; v_2 \; \text{---} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ -5 & -5 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & -3/5 \\ 0 & 1 & 1 & 2/5 \end{bmatrix}$$

We need to add two more rows at the bottom of the matrix

$$\begin{bmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \end{bmatrix}$$

to turn it into an invertible square matrix.

We do so by identifying the missing pivot 1's of the RREF matrix.
The pivot 1's of third and fourth rows are missing. Therefore, we choose $w_1 = (0,0,1,0)$ and $w_2 = (0,0,0,1)$.

$$\begin{bmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ \text{---} & w_1 & \text{---} \\ \text{---} & w_2 & \text{---} \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{invertible}}$$

The space W that we need to find is the space that has basis $\{w_1, w_2\}$.

Ex: Let V be a subspace of $\mathbb{R}^5$ that has basis $\mathcal{B} = \{v_1, v_2, v_3\}$.
Suppose after doing row reduction we get

$$\begin{bmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ \text{---} & v_3 & \text{---} \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find a subspace W of $\mathbb{R}^5$ such that $V \oplus W = \mathbb{R}^5$.
We look for the missing pivot 1's in the RREF matrix. We see that there are no pivot 1's on the second and fourth column. Thus, we will choose

$$w_1 = (0,1,0,0,0),$$
$$w_2 = (0,0,0,1,0).$$

Then $W = \operatorname{span}\{w_1, w_2\}$ is the space we look for.

Ex: See the worksheet for a similar example where V is an abstract vector space.

* Sum of many vector spaces:

Following the same idea of how to define $U+V$, we can define the sum of vector spaces $V_1, V_2, \dots, V_n$ as

$$V_1+V_2+\dots+V_n = \{v_1+v_2+\dots+v_n : v_i \in V_i \text{ for } i=1,2,\dots,n\}.$$

This is the smallest vector space that contains all of $V_1, V_2, \dots, V_n$

How to find a basis of $V_1+V_2+\dots+V_n$?

- Find a basis B_1 of V_1, B_2 of V_2, ..., B_n of V_n.
- Concatenate them: $B_1 \sqcup B_2 \sqcup \dots \sqcup B_n$ This is now a spanning set of $V_1+V_2+\dots+V_n$ We then need to eliminate the "redundant" vectors.
- Arrange these vectors as columns of a matrix
- Do row reduction. The pivot columns will tell us which vectors in $B_1 \sqcup B_2 \sqcup \dots \sqcup B_n$ to take in order to form a basis.

The sum $V_1+\dots+V_n$ is called a direct sum if one of the following conditions is met:

1) $B_1 \sqcup B_2 \sqcup \dots \sqcup B_n$ is linearly independent
2) $\dim(V_1+\dots+V_n) = \dim V_1+\dots+\dim V_n$.
3) For each $i=1,2,\dots,n$ we have
$$V_i \cap (V_1+\dots+V_{i-1}+V_{i+1}+\dots+V_n) = \{0\}.$$
4) Each $x \in V_1+\dots+V_n$ can be written only in one way as $x = v_1+\dots+v_n$ where $v_1 \in V_1$, $v_2 \in V_2, \dots, v_n \in V_n$

If $V_1+\dots+V_n$ is a direct sum, we can write as $V_1 \oplus V_2 \oplus \dots \oplus V_n$