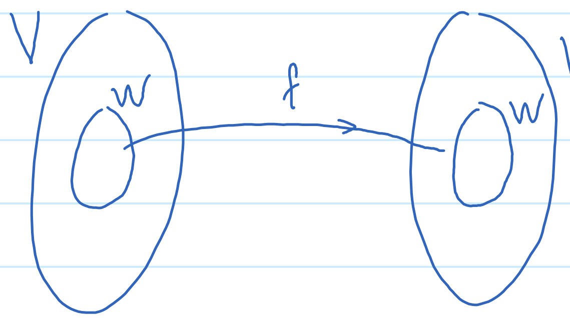
Lecture 12

Monday, February 3, 2020

Let V be a vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $f: V \to V$ be a linear map. A subspace W of V is said to be invariant under f if $f(W) \subset W$. Here $f(W)$ denotes the image of W under f.

$$f(W) = \{ f(v) : v \in W \}$$



In other words, W is said to be invariant under f if every vector in W gets mapped to a vector in W.

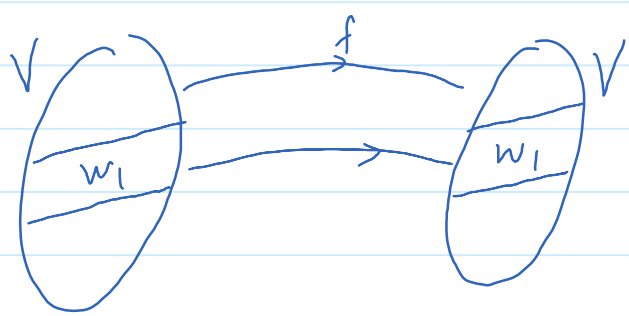
The concept of "invariance" is in fact quite general, not limited in the context of linear maps and vector spaces. Consider the following example.

$V = \{$students between 10-16 years of age$\}$

$f: V \to V$ mapping a student to his/her favorite classmate.

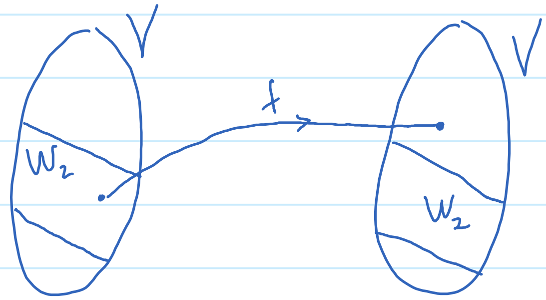
$W_1 = \{$students of age 12$\}$,

$W_2 = \{$students who like math$\}$

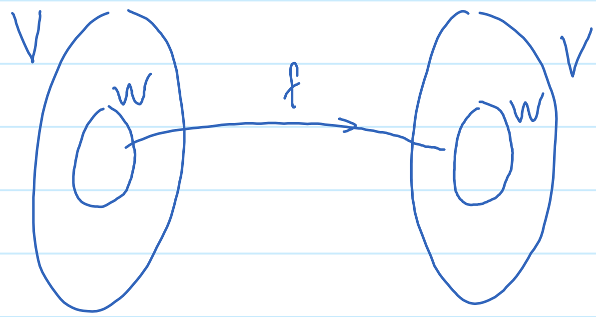


Each student of age 12 has a favorite classmate (of the same age). This way, f takes an element in W_1 to another element in W_1. Thus, W_1 is invariant under f.

On the other hand, the favorite classmate of a student who likes math doesn't necessarily like math In other word, f can map an element of W_2 to an element outside of W_2. Therefore, W_2 is not invariant under f.



Note that invariance is a relative property. A set may be invariant under a certain map but not other maps. In linear algebra, invariant subspaces help us "localize" a linear map to a smaller vector space. If $W \subset V$ is an invariant subspace then the restriction of f on W is also a linear map (going from W to W).



This localization helps simplify the linear map. For example, if W is a one-dimensional invariant subspace of V then f maps a vector w in W to a scalar multiple of w. In other words, f doesn't rotate any vector on W. Invariant subspaces are important in spectral theory, in which we try to decompose a big vector space into many small (1-dimensional) invariant subspaces. We will discuss in more detail when we learn eigenvalues and eigenspaces.

<u>Ex</u>: Consider a linear map $f: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $f(A) = A^T$. Consider two subspaces of $M_{2\times 2}(\mathbb{R})$:

$$V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b+c=0 \right\},$$

$$V_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+2b=0 \right\}.$$

Which subspace is invariant under f?

- Let us consider V_1 first. A general element of V_1 is of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } b+c=0$$

The image of A under f is $\quad f(A) = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$

This is a vector in $M_{2\times2}(\mathbb{R})$ that belongs to V_1 because $c+b=0$

Therefore, f maps a matrix $A \in V_1$ to a matrix $f(A) = A^T \in V_1$.

We conclude that V_1 is invariant under f.

- Let us consider V_2. A general element of V_2 is of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } a+2b=0$$

The image of A under f is $\quad f(A) = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$

The condition for this matrix to be in V_2 is $a+2c=0$.

However, we only have $a+2b=0$. The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

satisfies $a+2b=0$ but doesn't satisfy $a+2c=0$. Thus, $A \in V_2$

but $f(A) \notin V_2$ We conclude that V_2 is <u>not</u> invariant under f

[See more examples on the worksheet]