Lecture 13

Wednesday, February 5, 2020

Last time, we defined invariant subspace of a vector space with respect to a linear map. Consider a linear map $f: V \to V$. A subspace W of V is said to be invariant under f if $f(W) \subset W$.

A special invariant subspace of V is known as the eigenspace. Let $\lambda \in F$, which is the base field of V. The set

$$E_\lambda = \{v \in V : f(v) = \lambda v\}$$

consists of all vectors in V that get scaled by a factor of λ when applying f.

Ex: Show that E_λ is

(a) a subspace of V,

(b) invariant under f.

The proof of (a) follows from the standard procedure: to check if E_λ is a subspace of V, knowing that it is already a subset of V, we only need to check 3 properties:

1) $0 \in E_\lambda$

2) E_λ is closed under addition.

3) E_λ is closed under scaling

How to check 2)? Let $v, w \in E_\lambda$. We want to show $v + w \in E_\lambda$. Because $v, w \in V$, we have

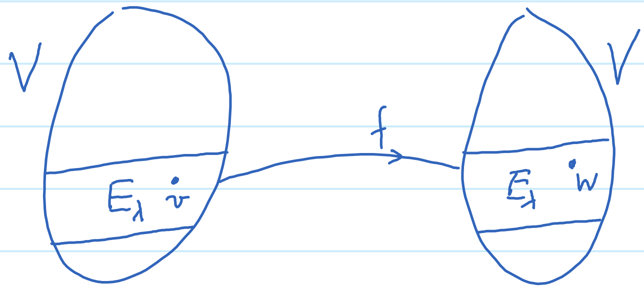
$$f(v) = \lambda v,$$
$$f(w) = \lambda w.$$

To show that $v + w \in E_\lambda$, we need to show $f(v+w) = \lambda(v+w)$. We have

$$\begin{aligned} \text{LHS} &= f(v) + f(w) && \text{(because } f \text{ is linear)} \\ &= \lambda v + \lambda w && \text{(because } v, w \in E_\lambda) \\ &= \lambda(v+w) \\ &= \text{RHS}. \end{aligned}$$

Thus, we have proved 2). Why is 1) true? Notice that $f(0) = 0 = \lambda 0$. Thus, $0 \in E_\lambda$. Property 3) can be proven in the same manner at Property 2).

(b) How to show that E_λ is invariant under f?
Let $v \in E_\lambda$. We show that $f(v) \in E_\lambda$.



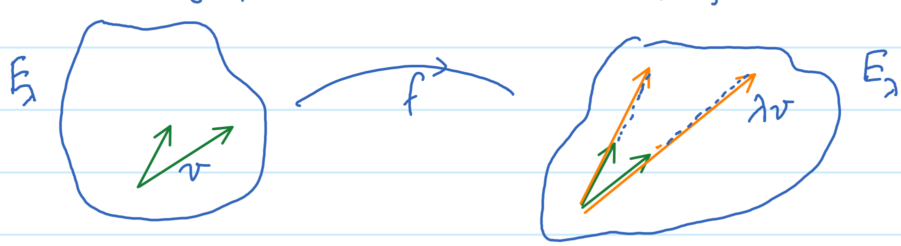
Put $w = f(v)$. To show that $w \in E_\lambda$, we need to show $f(w) = \lambda w$.
Because $v \in E_\lambda$, we have $f(v) = \lambda v$. In other words, $w = \lambda v$.
We have

$$f(w) = f(\lambda v) = \lambda f(v) = \lambda w$$

which is what we wanted to show. ■

For most λ, E_λ is equal to $\{0\}$, the trivial vector space. There are some very special values λ such that $E_\lambda \neq \{0\}$. Those values are called eigenvalues of f. If λ is an eigenvalue of f then E_λ is called eigenspace corresponding to λ. A nonzero element of E_λ is called an eigenvector corresponding to λ.

In other words, an eigenvector of f corresponding to λ is a vector that gets scaled by factor λ when applying f.



An eigenvector of f is not rotated by f, only scaled. In E_λ, every vector is scaled by factor λ.

The notion of eigenvalue and eigenvectors of a linear map is a generalization of the notion of eigenvalues and eigenvectors of a matrix, which we already learned in Linear Algebra I. The connection between them is a follows.

If $A \in M_{n\times n}(F)$ then the eigenvalues and eigenvectors of A are exactly the eigenvalues and eigenvectors of the linear map $f: \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = Av$.

Why so? If λ is an eigenvalue of A then there is some $v \neq 0$ such that $(A - \lambda I_n)v = 0$. This is equivalent to $Av - \lambda I_n v = 0$, or simply $Av = \lambda v$. Thus, $f(v) = \lambda v$. This means $v \in E_\lambda$ and $v \neq 0$. Then λ is an eigenvalue of f. One can also show that the converse is true: an eigenvalue of f is also an eigenvalue of A.

More generally, we have the following:
Let $f: V \to V$ be a linear map. Let $\mathcal{B}$ be a basis of V, and $A = [f]_{\mathcal{B}}$ be the matrix that represents f in basis $\mathcal{B}$. Then the eigenvalues of f are the same as the eigenvalues of A and vice versa.

How do we see this? Let $\lambda \in F$ be an eigenvalue of f. By the definition of eigenvalues, we have $E_\lambda \neq \{0\}$. This means there is a nonzero vector $v \in V$ such that

$$f(v) = \lambda v.$$

Let us take the coordinate of both sides in basis $\mathcal{B}$.

$$[f(v)]_{\mathcal{B}} = [\lambda v]_{\mathcal{B}}$$

We know that

$$\text{LHS} = [f(v)]_{\mathcal{B}} = [f]_{\mathcal{B}} [v]_{\mathcal{B}} = A[v]_{\mathcal{B}},$$
$$\text{RHS} = [\lambda v]_{\mathcal{B}} = \lambda [v]_{\mathcal{B}}.$$

Hence, we obtain an equation

$$A[v]_{\mathcal{B}} = \lambda [v]_{\mathcal{B}}. \qquad (*)$$

Note that $[v]_{\mathcal{B}} \neq 0$ because $v \neq 0$. From $(*)$, we realize that $[v]_{\mathcal{B}}$ is an eigenvector of A and λ is the corresponding eigenvalue. We have showed that an eigenvalue of f is an eigenvalue of A. The converse can be shown similarly: an eigenvalue of A is also an eigenvalue of f

We have found a connection between the eigenvalues of f and the eigenvalues of the matrix that represents f: they are the same. How about the eigenspace E_λ of f and the eigenspace $\widetilde{E}_\lambda$ of $A = [f]_{\mathcal{B}}$? What is the relation between them? We know that

$$E_\lambda = \{v \in V : f(v) = \lambda v\},$$
$$\widetilde{E}_\lambda = \{x \in \mathbb{R}^n : Ax = \lambda x\}. \quad (\text{Here } n = \dim V.)$$

From $(*)$, we have

$$\widetilde{E}_\lambda = \{[v]_{\mathcal{B}} : v \in E_\lambda\}$$

In other words, $\widetilde{E}_\lambda$ is the set of all the coordinates (in basis $\mathcal{B}$) of the eigenvectors of f.

This observation gives us a method to find the eigenvalues and eigenvectors of f by finding the eigenvalues and eigenvectors of the matrix $A = [f]_{\mathcal{B}}$. The procedure is as follows

1) Given a function $f: V \to V$, fix a basis $\mathcal{B}$ of V and find the matrix $[f]_{\mathcal{B}}$. Call it A.
2) Find the eigenvalue λ's and the corresponding eigenspaces $\widetilde{E}_\lambda$ of A. We know how to do this in Linear Algebra I.
3) For each eigenvalue λ, find a basis of $\widetilde{E}_\lambda$.
4) Then convert this basis to a basis of E_λ.

Ex: Let $f: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear map given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of f.

Fix a standard basis of $M_{2\times 2}(\mathbb{R})$:

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$$

We have

$$f(E_1) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = E_4$$

$$f(E_2) = f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -E_2$$

$$f(E_3) = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = -E_3$$

$$f(E_4) = f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = E_1$$

Thus, the matrix that represents f in basis $\mathcal{B}$ is

$$[f]_{\mathcal{B}} = \begin{bmatrix} | & | & | & | \\ [f(E_1)]_{\mathcal{B}} & [f(E_2)]_{\mathcal{B}} & [f(E_3)]_{\mathcal{B}} & [f(E_4)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{A}$$

What are the eigenvalues of A?

$$\det(A-\lambda I_4) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \ldots$$

$$= (\lambda+1)^3(\lambda-1).$$

One can use Matlab to compute the eigenvalues of A as follows:

```
>> A = [0 0 0 1; 0 -1 0 0; 0 0 -1 0; 1 0 0 0]
>> eig(A)
```

A has two eigenvalues: $\lambda = \pm 1$.

- Find eigenspace $\widetilde{E}_1$ (corresponding to $\lambda = 1$):

To find $\widetilde{E}_1$, we find all vectors $x \in \mathbb{R}^4$ that satisfy

$$Ax = 1.x$$

This equation is equivalent to $(A - I_4)x = 0$

We are finding the null space of matrix $A - I_4$.

$$A - I_4 = \begin{array}{c} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array} \xrightarrow{RREF} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array}$$

We get $x_1 = x_4$, $x_2 = x_3 = 0$ Thus,

$$\widetilde{E}_1 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 = x_4,\ x_2 = x_3 = 0\}$$
$$= \{(x_1,0,0,x_1) : x_1 \in \mathbb{R}\}$$
$$= \operatorname{span}\{(1,0,0,1)\}.$$

Thus, $\widetilde{E}_1$ has basis $\{(1,0,0,1)\}$.

- Find eigenspace $\widetilde{E}_{-1}$ of A:

We get $\widetilde{E}_{-1} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 = -x_4\}$

$$= \{(x_1, x_2, x_3, -x_1) : x_1, x_2, x_3 \in \mathbb{R}\}$$
$$= \operatorname{span}\{(1,0,0,-1), (0,1,0,0), (0,0,1,0)\}.$$

One can use Matlab to assist the computation of the eigenvalues and eigenvectors of A as follows:

```
>> A = [0 0 0 1; 0 -1 0 0; 0 0 -1 0; 1 0 0 0]
>> [a, b] = eig(A)
```

Matlab returns:

$$a = \begin{bmatrix} 0 & 0 & 0.7071 & 0.7071 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -0.7071 & 0.7071 \end{bmatrix}$$

$$b = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix b is a diagonal matrix that contains the eigenvalues of A on the diagonal. Matrix a consists of eigenvectors of A (aligned a columns) corresponding to the eigenvalues in b For example, the vector

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

which is the first column of a is an eigenvector of $\lambda = -1$.

Vectors $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0.7071 \\ 0 \\ 0 \\ -0.7071 \end{bmatrix}$

are also eigenvectors of $\lambda = -1$. Why does the entries of v_3 look so ugly? This is because Matlab tried to rescale eigenvector to be of magnitude 1. The magnitude of v_3 is

$$\sqrt{(0.7071)^2 + 0^2 + 0^2 + (-0.7071)^2} = 1$$

One can replace v_3 by $v_3' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

which is a nicer scaling of v_3'. Continue next time.