Lecture 14

Friday, February 7, 2020

Continue the example last time:

$$f: M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Find the eigenvalues and eigenvectors of f.

We translated the problem of finding eigenvalues and eigenvectors of f to the problem of finding the eigenvalues and eigenvectors of the matrix

$$A = [f]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We found that A has two eigenvalues $\lambda = 1$ and $\lambda = -1$.

The eigenspace of A corresponding to $\lambda = 1$ is

$$\tilde{E}_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The eigenspace of A corresponding to $\lambda = -1$ is

$$\tilde{E}_{-1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Now we translate back to f: the eigenvalues of f are $\lambda = \pm 1$.

The eigenspace of f corresponding to $\lambda = 1$ is

$$E_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the vector in $M_{2\times 2}(\mathbb{R})$ that has coordinate $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ with respect to basis $\mathcal{B}$.

Similarly, the eigenspace of f corresponding to $\lambda = -1$ is

$$E_{-1} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

We have found the eigenvalues and eigenspaces of f. Our method was to translate the problem on the abstract vector space $M_{2\times 2}(\mathbb{R})$ to a concrete vector space $\mathbb{R}^4$. This is done by fixing a basis of $M_{2\times 2}(\mathbb{R})$ (we chose the standard basis) and replace abstract vectors (which are matrices in this case) by their coordinate vectors (vectors in $\mathbb{R}^4$).

There is another method to find the eigenvalues and eigenspaces of f which doesn't resort to coordinates. We will discuss it after the midterm.