Lecture 15

Wednesday, February 12, 2020

Consider the linear map $f: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Last time, we found the eigenvalues and eigenspaces of f by using coordinates. In particular, instead of finding the eigenvalues and eigenspaces of f, we found the eigenvalues and eigenspaces of matrix $A = [f]_{\mathcal{B}}$. Then convert the result back to f. This is an indirect method. It is quite general and systematic. One can use it for any problem. The drawback is that it often requires a lot of computations.

For example, we had to find eigenvalues / eigenspaces of a 4×4 matrix. If f were a linear map from $M_{3\times 3}(\mathbb{R})$ to $M_{3\times 3}(\mathbb{R})$, we would have to deal with a 9×9 matrix. The large size of matrix A costs more calculations. Now we will use a coordinate-free method. This is a direct method, coming straightly from the definition of eigenvalues.

We know that λ is called an eigenvalue of f if the space

$$E_\lambda = \{v \in V : f(v) = \lambda v\}$$

is bigger than $\{0\}$. In other words, λ is an eigenvalue of f if one can find a vector $v \in V$, $v \neq 0$ such that $f(v) = \lambda v$.

For this reason, we will solve the equation

$$f(v) = \lambda v$$

for two unknowns $\lambda \in F$ and $v \in V$, $v \neq 0$.

We call this method a direct method. At the first glance, it seems odd to solve <u>one equation</u> for <u>two unknowns</u>. One can expect that there are infinitely many solutions λ and v. However, we will see that for almost all λ, vector v has to be zero, which is not acceptable. Only a few values of λ can give a nonzero vector v. These are the eigenvalues

Eigenvalues and eigenvectors are special (and helpful) pieces of information about a linear map. They are hidden in the map and usually require a considerable amount of computation to find. Once we find them, we can use them to better understand the linear map.

Let us go back to the problem We want to solve for $\lambda \in \mathbb{R}$ and $v \in M_{2\times 2}(\mathbb{R})$, $v \neq 0$ such that

$$f(v) = \lambda v.$$

Write $\quad v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $\quad f(v) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

We want to solve for $\lambda \in \mathbb{R}$, and $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from the equation

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This equation is equivalent to

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

This is equivalent to solving a system of 4 equations:

$$(I): \begin{cases} d = \lambda a & (1) \\ -b = \lambda b & (2) \\ -c = \lambda c & (3) \\ a = \lambda d & (4) \end{cases}$$

From the second and third equation, we get

$$(1+\lambda) b = 0, \qquad (2')$$

$$(1+\lambda) c = 0. \qquad (3')$$

From the first and fourth equation, we get

$$(\lambda^2 - 1) d = 0 \qquad (1')$$

From equations $(1'), (2'), (3')$, we observe that for most values of λ, we get $b=c=d=0$ (except when $\lambda=-1$ or $\lambda=1$). Then a is also equal to 0 because $a=\lambda d$. The case $\lambda \neq \pm 1$ gives a trivial vector

$$v=\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is not what we are looking for. In other words, any λ which is neither -1 nor 1 is <u>not</u> an eigenvalue of f. There are only two candidates for eigenvalues. $\lambda=1$ and $\lambda=-1$ Let us examine them in detail.

- $\lambda = 1$:

System (I) becomes

$$\begin{cases} d = a \\ -b = b \\ -c = c \\ a = d \end{cases}$$

which gives $b=c=0$, $a=d$. There is no constraint on a and d other than $a=d$. We get

$$v=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

which is nonzero if a is chosen to be nonzero. Therefore, $\lambda=1$ is indeed an eigenvalue of f. The corresponding eigenspace is

$$E_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- $\lambda = -1$:

System (I) becomes

$$\begin{cases} d = -a \\ -b = -b \\ -c = -c \\ a = -d \end{cases} \qquad \text{which is equivalent to } a=-d.$$

There is no constraint on b and c. Therefore,

$$v = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where a, b, c are free. This matrix is of course nonzero if one of a, b, c is nonzero. We conclude that $\lambda = -1$ is indeed an eigenvalue of f. The corresponding eigenspace of f is

$$E_{-1} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

* <u>Diagonalizable linear maps</u>:

A linear map $f: V \to V$, where V is a vector space over F, is said to be **diagonalizable** if <u>one of the following</u> is true:

1) V can be decomposed into 1-dimensional invariant subspaces:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

2) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of f then

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

3) There is a basis $B = \{v_1, v_2, \ldots, v_n\}$ of V such that each v_i is an eigenvector of f

Statements 1), 2), 3) are equivalent. If one is true, so are the others. If one is false, so are the others. We will consider some examples next time