Lecture 16

Friday, February 14, 2020

In Linear Algebra I, we learned diagonalizable matrices: matrix A is said to be diagonalizable if there is a diagonal matrix D and an invertible matrix P such that $A = P^{-1}DP$.

Most matrices are diagonalizable! To see why, let us consider a 2×2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The characteristic polynomial of A is
$$\begin{aligned} f(x) &= \det(A - xI_2) \\ &= \begin{vmatrix} a-x & b \\ c & d-x \end{vmatrix} \\ &= x^2 - (a+d)x + ad - bc. \end{aligned}$$

This is a quadratic polynomial. Although it may not have a real root, it always has complex roots. It has two distinct complex roots if
$$\Delta = (a+d)^2 - 4(ad-bc) \neq 0.$$

In this case, the matrix is diagonalizable because each root is a simple root. The only case where A might not be diagonalizable is when $\Delta = 0$. This is when
$$(a+d)^2 - 4(ad-bc) = 0. \qquad (*)$$

For a randomly chosen matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the equation $(*)$ is almost never satisfied. Thus, it is almost surely that a randomly chosen matrix is diagonalizable. Non-diagonalizable matrices are rare!

Matrix diagonalization is a special case of the diagonalization of linear maps. Most linear maps are diagonalizable. Recall the definition:

A linear map $f: V \to V$, where V is a vector space over F, is said to be diagonalizable if one of the following is true:

1) V can be decomposed into 1-dimensional invariant subspaces:
$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$
2) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of f then
$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$
3) There is a basis $B = \{v_1, v_2, \ldots, v_n\}$ of V such that each v_i is an eigenvector of f
4) V has a basis B such that $[f]_B$ is a diagonal matrix.

These four statements are equivalent to each other. Statement 2) is what we usually use to check if a map is diagonalizable The procedure is as follows:

- Compute all the distinct eigenvalues of f. Label them as $\lambda_1, \lambda_2, \ldots, \lambda_k$.
- To each eigenvalue λ_j, find a basis of the corresponding eigenspace E_{λ_j}.
- If $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = V$ then f is diagonalizable. Otherwise, f is not diagonalizable.

The most troublesome step of this procedure is perhaps how to check if $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = V$. Technically, one would have to check two things:
$$\begin{cases} \text{the sum } E_{\lambda_1} + \cdots + E_{\lambda_k} \text{ is a direct sum,} & (*) \\ E_{\lambda_1} + \cdots + E_{\lambda_k} = V & (**) \end{cases}$$
However, we will see that the only thing we need to check is whether $\dim E_{\lambda_1} + \cdots + \dim E_{\lambda_k} = \dim V$. This is thanks to the following theorem:

Theorem:

If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of an arbitrarily linear map $f: V \to V$ (not necessarily diagonalizable), then the sum $E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum.

For this reason, $(*)$ is always true for any linear map. We don't nee to check it. What we need to check is whether $E_{\lambda_1} + \dots + E_{\lambda_k} = V$. Note that the sum $E_{\lambda_1} + \dots + E_{\lambda_k}$ is already a subspace of V. To check if it is equal to V, we only need to check if

$$\dim(E_{\lambda_1} + \dots + E_{\lambda_k}) = \dim V.$$

Because the sum $E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum (the above theorem), we have

$$\dim(E_{\lambda_1} + \dots + E_{\lambda_k}) = \dim E_{\lambda_1} + \dim E_{\lambda_2} + \dots + \dim E_{\lambda_k}.$$

Therefore, all we need to check is if

$$\dim E_{\lambda_1} + \dots + \dim E_{\lambda_k} = \dim V$$

In conclusion, the procedure to check if a map $f: V \to V$ is diagonalizable can be simplified as follows:

- Compute all the distinct eigenvalues of f. Label them as $\lambda_1, \lambda_2, \dots, \lambda_k$.
- To each eigenvalue λ_j, find a basis of the corresponding eigenspace E_{λ_j}.
- If $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dots + \dim E_{\lambda_k} = \dim V$ then f is diagonalizable.
  If $\dim E_{\lambda_1} + \dots + \dim E_{\lambda_k} < \dim V$ then f is not diagonalizable

Ex: Consider a linear map $G: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by
$$G(u)(x) = u(x+1).$$

Is G diagonalizable?

First we find the eigenvalues and eigenspaces of G. As we know, there are two ways to do this: indirect method (using coordinate), and direct method (coordinate free). Let us use the direct method.

We solve for $\lambda \in \mathbb{R}$ and $u \in P_2(\mathbb{R})$, $u \neq 0$ from the equation

$$G(u) = \lambda u.$$

Write $u(x) = ax^2 + bx + c$. Then
$$\begin{aligned} G(u) &= u(x+1) \\ &= a(x+1)^2 + b(x+1) + c \\ &= ax^2 + (2a+b)x + a+b+c. \end{aligned}$$

The equation $G(u) = \lambda u$ becomes

$$ax^2 + (2a+b)x + (a+b+c) = \lambda a x^2 + \lambda b x + \lambda c \quad \forall x \in \mathbb{R}$$

This is equivalent to

$$\begin{cases} a = \lambda a, \\ 2a + b = \lambda b, \\ a+b+c = \lambda c. \end{cases}$$

We try to factor out as much as we can:

$$\begin{cases} a(\lambda - 1) = 0 \\ b(\lambda - 1) = 2a \\ c(\lambda - 1) = a + b \end{cases} \qquad (I)$$

If $\lambda \neq 1$ then the first equation gives $a = 0$. Then the second equation gives $b = 0$. Then the third equation gives $c = 0$. Therefore, $a = b = c = 0$ if $\lambda \neq 1$. We conclude that any $\lambda \neq 1$ is <u>not</u> an eigenvalue of G.

Let us check if $\lambda = 1$ is an eigenvalue of G. The system (I) becomes

$$\begin{cases} a0 = 0, \\ b0 = 2a, \\ c0 = a+b. \end{cases}$$

From this system, we get $a = b = 0$, but c is arbitrary. Therefore, $\lambda = 1$ is indeed an eigenvalue of G. It is the only eigenvalue of G. The eigenspace is

$$E_1 = \{u = ax^2+bx+c : a=b=0\}$$
$$= \{u = c : c \in \mathbb{R}\}$$
$$= \operatorname{span}\{1\}.$$

The dimension of E_1 is $\dim E_1 = 1 < \dim \mathcal{P}_2(\mathbb{R}) = 3$.

We conclude that G is not diagonalizable.