Lecture 18

Wednesday, February 19, 2020

The addition and scaling operator of a general vector space are not sufficient to capture the idea of angles, which is a natural geometric of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ in general. To define angles, one needs to enrich the structure of a vector space. It turns out that we need something similar to the dot product in $\mathbb{R}^n$. Inner product is a generalization of dot product to a general vector space. Recall the definition of inner product:

Let V be a vector space over F. A binary operator on V, taking value on F, is an inner product if it satisfies the following axioms:

(1) Linearity in the first argument:

$$(u+v, w) = (u,w) + (v,w) \quad \forall u, v, w \in V,$$

$$(cu, w) = c(u,w) \quad \forall u, w \in V,\ c \in F.$$

(2) Conjugate symmetry:

$$(u,v) = \overline{(v,u)} \quad \forall u, v \in V.$$

complex conjugate

(3) Positive definiteness:

$$(u,u) \geq 0 \quad \forall u \in V.$$

If $(u,u) = 0$ then $u = 0$.

A vector space equipped with an inner product is called an inner product space.

One may ask: how about linearity in the second argument? Is it true? We in fact have the following:

$$(u, v+w) = (u,v) + (u,w) \quad \forall u, v, w \in V,$$

$$(u, cw) = \bar{c}(u,w) \quad \forall u, w \in V,\ c \in F.$$

complex conjugate

Why is this true?

$$\begin{aligned}(u, v+w) &= \overline{(v+w, u)} && \text{(conjugate symmetry)}\\ &= \overline{(v,u) + (w,u)} && \text{(linearity in the first argument)}\\ &= \overline{(v,u)} + \overline{(w,u)}\\ &= (u,v) + (u,w).\end{aligned}$$

We see that the inner product is additive in the second argument. On the other hand,

$$(u, cv) = \overline{(cv, u)} = \overline{c(v,u)} = \bar{c}\,\overline{(v,u)} = \bar{c}\,(u,v).$$

We see that the inner product is almost scalar multiplicative on the second argument. When we factor a constant outside of the inner product, we have to take the conjugate of the constant factor.

An operator $(\cdot,\cdot)$ that satisfies the axioms (1) and (2) is called a sesquilinear form. The prefix sesqui means $1\frac{1}{2}$. If the operator is linear in first and second argument, it would be called a bilinear form.

If $F = \mathbb{R}$ then an inner product on V is a bilinear form. This is because $\bar{c} = c$ for any $c \in \mathbb{R}$.

Ex: Let V be an inner product space. Show that $(0,0) = 0$.

By the linearity on the first argument:

$$(0,0) = (0+0, 0) = (0,0) + (0,0)$$

Thus, $(0,0) = 0$.

Ex: On $\mathbb{C}$, consider the operator $(\cdot,\cdot)$ given by

$$(z, w) = z\bar{w} \qquad \forall z, w \in \mathbb{C}.$$

Show that $(\cdot,\cdot)$ is an inner product on $\mathbb{C}$.

We need to check all the axioms.

* Check linearity on the first argument:

Let $z, w, v \in \mathbb{C}$. We have

$$(z+v, w) = (z+v)\bar{w} = z\bar{w} + v\bar{w} \quad \text{(distribution rule)}$$
$$= (z,w) + (v,w)$$

Let $z, w \in \mathbb{C}$ and $c \in \mathbb{C}$. We have

$$(cz, w) = cz\bar{w} = c(z,w).$$

* Check conjugate symmetry:

Let $z, w \in \mathbb{C}$. We have

$$(z, w) = z\bar{w} = \overline{\bar{z}w} = \overline{w\bar{z}} = \overline{(w,z)}.$$

* Check positive definiteness:

Let $z \in \mathbb{C}$. We have

$$(z,z) = z\bar{z} = |z|^2 \geq 0.$$

Recall that if $z = a+ib$ $(a, b \in \mathbb{R})$ then $z\bar{z} = (a+ib)(a-ib)$
$= a^2+b^2 = |z|^2$.

If $(z,z) = 0$ then $a^2+b^2 = 0$. This happens only if $a=b=0$, which means $z=0$.

Ex: On $\mathbb{C}$, consider an operator $(z, w) = zw$.

Show that $(\cdot,\cdot)$ is not an inner product.

To show that $(\cdot,\cdot)$ is not an inner product, we only need to give a counterexample showing how one of the axioms is violated.

Let us take $z = i$. Then

$$(z,z) = (i,i) = i^2 = -1 < 0.$$

Thus the positive definiteness condition is violated.

* Norm induced by an inner product:

Let V be an inner product space. The operator $\|.\|$ given by

$$\|v\| = \sqrt{(v,v)}$$

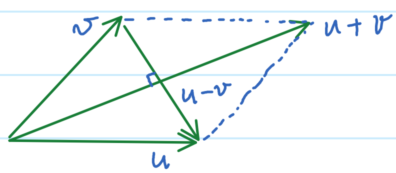
is called the norm on V induced by (or associated with) the inner product.

Norm of a vector is also called length or magnitude of the vector.

Ex:

Let V be a real inner product space (i.e. $F=\mathbb{R}$) and $u, v \in V$ be of equal length Show that $u+v$ and $u-v$ are perpendicular to each other.

[Two vectors called perpendicular or orthogonal to each other if their inner product is equal to zero.]



We have $\|u\| = \|v\|$ Thus, $\sqrt{(u,u)} = \sqrt{(v,v)}$. We get $(u,u)=(v,v)$.

We want to show $(u+v,\ u-v) = 0$.

We have

$$(u+v,\ u-v) = (u,\ u-v) + (v,\ u-v)$$
$$= (u,u) - (u,v) + (v,u) - (v,v)$$

Because $(u,u) = (v,v)$, we get

$$(u+v,\ u-v) = -(u,v) + (v,u) \qquad (*)$$

By the conjugate symmetry property of inner product,

$$(u,v) = \overline{(v,u)} \qquad (**)$$

Since $F=\mathbb{R}$, $(v,u) \in \mathbb{R}$. Thus, $\overline{(v,u)} = (v,u)$. Then $(**)$ becomes

$$(u,v) = (v,u).$$

Applying this identity to $(*)$, we get $(u+v,\ u-v) = -(u,v) + (u,v) = 0$.

Therefore, $u+v$ and $u-v$ are orthogonal to each other.