Lecture 19

Friday, February 21, 2020

Let V be an inner product space. Let $u_1, u_2 \in V$ and $a, b, c, d \in F$. How do we expand the inner product $(au_1 + bu_2, cu_1 + du_2)$?
We have

$$(au_1 + bu_2, cu_1 + du_2) = (au_1, cu_1 + du_2) + (bu_2, cu_1 + du_2)$$

(linearity in the first argument)

$$= (au_1, cu_1) + (au_1, du_2) + (bu_2, cu_1) + (bu_2, du_2)$$

(additive in the second argument)

$$= a\bar{c}(u_1, u_1) + a\bar{d}(u_1, u_2) + b\bar{c}(u_2, u_1) + b\bar{d}(u_2, u_2).$$

An easy way to remember this is to regard the inner product as a "regular" product $(au_1 + bu_2)(cu_1 + du_2)$ although this product doesn't make sense. Then use distribution law:

$$ac\, u_1 u_1 + bc\, u_2 u_1 + ad\, u_1 u_2 + bd\, u_2 u_2 .$$

Each coefficient on the second factor has to come with complex conjugate:

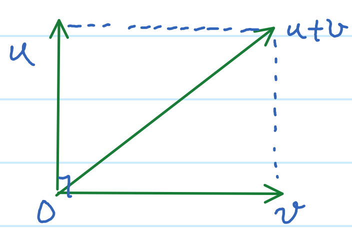
$$a\bar{c}\,(u_1, u_1) + b\bar{c}\,(u_2, u_1) + a\bar{d}\,(u_1, u_2) + b\bar{d}\,(u_2, u_2).$$

This is the formula one needs for Problem 3 of HW 5.

<u>Ex</u>: (Pythagorean theorem)

Let V be an inner product space. Let $u, v \in V$ be such that $(u, v) = 0$. Show that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2. \qquad (*)$$



We have

$$\text{LHS}(*) = (u+v, u+v)$$
$$= (u,u) + (v,u) + (u,v) + (v,v).$$

Because $u \perp v$, we have

$$(u,v) = (v,u) = 0.$$

Thus, $\text{LHS}(*) = (u,u) + (v,v) = \|u\|^2 + \|v\|^2 = \text{RHS}(*)$.

We know that inner product provides an additional structure to a vector space that helps us define angles and lengths (norms). There is another structure (weaker) that helps us define lengths, but not angles. That is norm.

* Definition:

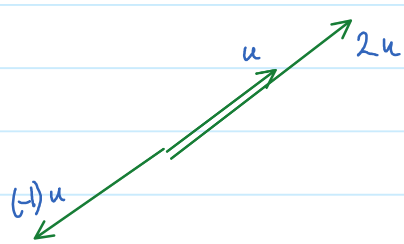
Let V be a vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. An operator $\|\cdot\| : V \to \mathbb{R}$ is said to be a norm on V if it satisfies three following axioms:

(1) Homogeneity:
$$\|cu\| = |c|\,\|u\| \quad \forall u \in V,\ c \in F.$$

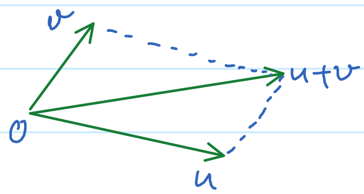
(2) Triangle inequality:
$$\|u+v\| \leq \|u\| + \|v\| \quad \forall u, v \in V.$$

(3) Positivity:
$$\|u\| \geq 0 \quad \forall u \in V.$$
If $\|u\| = 0$ then $u = 0$.

A vector space equipped with a norm is called a normed space.
Norm is a generalization of length in $\mathbb{R}^n$ to an abstract vector space. All of the axioms reflect basic properties of length.



The homogeneity property says that if we scale a vector by a factor c then the length of u is scaled by factor $|c|$.



The triangle inequality reflect a geometric inequality well-known in $\mathbb{R}^n$.

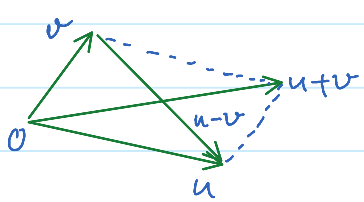
If V is an inner space then $\|v\| = \sqrt{(v,v)}$ defines a norm on V. One can easily check that this satisfies all the axioms of norm. This norm is a special type of norm in that it comes from an inner product.

$$\begin{tikzcd} \text{vector space} \arrow[r, "\text{inner product}"] \arrow[d, "\text{norm}"] & \text{inner product space (angles, lengths)} \\ \text{normed space (lengths)} & \end{tikzcd}$$

Not all norms come from an inner product. Indeed, the norm induced from an inner product has to satisfy a so-called parallelogram identity.

Ex: Let V be an inner product. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \forall u,v \in V.$$



The sum of the square of the diagonals is equal to the sum of the square of the edges.

Why is this true?

$$\|u+v\|^2 = (u+v, u+v) = (u,u) + (v,u) + (u,v) + (v,v).$$

$$\|u-v\|^2 = (u-v, u-v) = (u,u) - (v,u) - (u,v) + (v,v).$$

Sum these two equations:

$$\|u+v\|^2 + \|u-v\|^2 = 2(u,u) + 2(v,v) = 2\|u\|^2 + 2\|v\|^2.$$

Ex: $\mathbb{R}^2$ has a norm $\|x\| = |x_1| + |x_2|$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Why is this true?

We need to check every axiom of norm.

* Check homogeneity:

Let $c \in \mathbb{R}$ and $x \in \mathbb{R}^2$. We want to show that $\|cx\| = |c|\,\|x\|$.

Write $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $cx = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$.

Then
$$\|cx\| = |cx_1| + |cx_2| = |c|\,|x_1| + |c|\,|x_2| = |c|\,(|x_1| + |x_2|)$$
$$= |c|\,\|x\|.$$

* Check triangle inequality:

We know the triangle inequality of numbers:
$$|a+b| \le |a| + |b| \quad \forall a, b \in \mathbb{R}. \qquad (*)$$

Let $x, y \in \mathbb{R}^2$. We want to show $\|x+y\| \le \|x\| + \|y\|$.

We write $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Then $x+y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$.

Then
$$\begin{aligned} \|x+y\| &= |x_1 + y_1| + |x_2 + y_2| \\ &\le (|x_1| + |y_1|) + (|x_2| + |y_2|) \qquad (\text{due to } (*)) \\ &= (|x_1| + |x_2|) + (|y_1| + |y_2|) \\ &= \|x\| + \|y\|. \end{aligned}$$

* Check positivity: (can be done easily).

Ex: The norm $\|\cdot\|$ on $\mathbb{R}^2$ given by $\|x\| = |x_1| + |x_2|$ doesn't come from any inner product.

If this norm comes from an inner product then it must satisfy the parallelogram identity. Let us pick
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Then
$$x+y = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad x-y = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Then $\|x+y\| = |2|+|-1| = 3,$

$\|x-y\| = |0|+|5| = 5,$

$\|x\| = |1|+|2| = 3,$

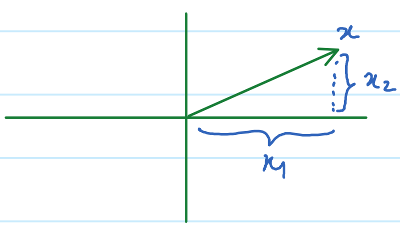
$\|y\| = |1|+|-3| = 4.$

We see that

$$\left.\begin{aligned} \|x+y\|^2+\|x-y\|^2 &= 3^2+5^2 = 34 \\ 2(\|x\|^2+\|y\|^2) &= 2(3^2+4^2) = 50. \end{aligned}\right\} \text{different.}$$

Because the parallel diagram is not satisfied, $\|\cdot\|$ doesn't come from an inner product.

This norm is known as the taxi cab norm. Imagine a taxi going from point O (the origin) to a point A on the plane. This point corresponds to vector



$$x = \overrightarrow{OA} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The taxi only travel on the roads, which only go horizontally or vertically. The shortest path to go from O to A is $|x_1|+|x_2|$.