Lecture 2

Wednesday, January 8, 2020

Consider
- a set V,
- a field of numbers $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$,
- an intrinsic addition on V, i.e a way to add two elements of V giving an element in V,

$$\underbrace{v}_{V} + \underbrace{w}_{V} \in V$$

- an extrinsic multiplication (also called scaling or scalar multiplication), i.e. a way to multiply an element of V by a number in F, giving an element in V.

$$\underbrace{c}_{F}\underbrace{v}_{V} \in V$$

The structure of these four ingredients $(V, F, +, \cdot)$ is called a vector space if the following properties are satisfied.

(A) Addition

(A0) $v + w \in V \quad \forall v, w \in V$

(V is closed under addition)

(A1) $v + w = w + v \quad \forall v, w \in V$

(the addition is commutative)

(A2) $u + (v + w) = (u + v) + w \quad \forall u, v, w \in V$

(the addition is associative)

(A3) There is an element $a \in V$ such that

$$a + v = v \quad \forall v \in V$$

(Later we'll show that such an element is unique. It is called the neutral element, or the zero element, denoted by 0.)

(A4) For each $v \in V$, there is an element $w \in V$ (called additive inverse of v) such that $v + w = a$

(S) scaling

(S0) $cv \in V \quad \forall c \in F,\ v \in V$

(V is closed under scaling)

(S1) $1v = v \quad \forall v \in V$

(1 is a multiplicative unit)

(S2) $c(dv) = (cd)v \quad \forall c, d \in F,\ v \in V$

(scaling is associative)

Note that the left hand side and the right hand side come from different processes.

LHS: $v \longrightarrow \boxed{\text{scale by } d} \xrightarrow{dv} \boxed{\text{scale by } c} \longrightarrow c(dv)$

(scale v twice)

RHS. $v \longrightarrow \boxed{\text{scale by } cd} \longrightarrow (cd)v$

(scale v only once)

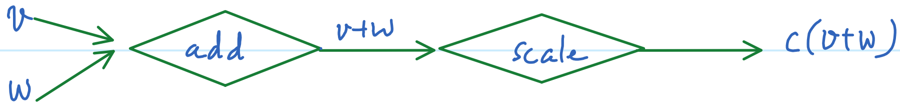
(S2) says that the results obtained from these two processes are the same.

(D) Distrubution between addition and scaling:

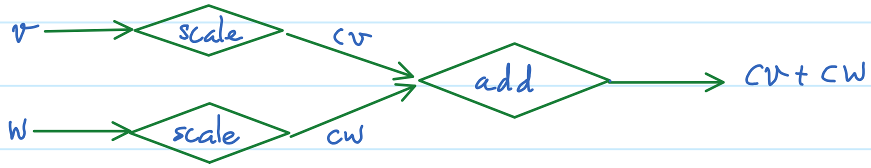
(D1) $\underbrace{c(v+w)}_{\text{add first, then scale}} = \underbrace{cv + cw}_{\text{scale each, then add}} \quad \forall c \in F,\ v, w \in V.$

Note that LHS and RHS are computed through different processes.

LHS:



RHS:



(D2) $(c+d)v = cv + dv \quad \forall c, d \in F, v \in V.$

A structure $(V, F, +, .)$ satisfying 10 properties in groups (A), (S), (D) is called a vector space. If the operations (addition and scaling) are well understood, we simply say that V is a vector space over F.

An element of V is called a vector. The neutral element a in the definition is called the zero vector, denoted by 0.

* Example of vector spaces:

① $\mathbb{R}^n$ is a vector space over $\mathbb{R}$,
$\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$,
$\mathbb{C}^n$ is a vector space over $\mathbb{C}$.
[F^n is a vector space over F.]

② $V = M_{m \times n}(F)$, the set of all matrices of size $m \times n$ with coefficients in F, is a vector space over F.
Here the addition is the regular addition of two matrices, and the scaling is the regular scaling of matrices. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

(3) $V = F^S$, the set of all function from S to F, is a vector space over field F, where the addition and scaling of functions are defined as follows:

With $f, g \in V$, i.e. functions from S to F, the function $f+g$ is defined as

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S.$$

With $c \in F$ and $f \in V$, the function cf is defined as

$$(cf)(x) = cf(x) \quad \forall x \in S$$

In other words, to add two functions we simply add their values at each point x. To scale at function, we simple scale its value at each point x.

[See practice on the worksheet.]