Lecture 20

Monday, February 24, 2020

We have covered all abstract axiomatic definitions in this course:

- Vector space,
- Linear map,
- Inner product,
- Norm.

$$\begin{tikzcd}
\text{Vector space } (+,\cdot) \arrow[loop left, "\text{linear map}"] \arrow[r, "\text{inner product}", "(\cdot,\cdot)"'] \arrow[d, "\text{norm}"', "\|\cdot\|"] & \text{Inner product space (angles, lengths)} \\
\text{normed space} &
\end{tikzcd}$$

The inner product turns out to be a very helpful notion. It helps us deal with optimization problems which will be discussed in detail later. Let us consider a few examples of inner products.

Ex: On $\mathbb{R}^n$, consider $(x,y) = x_1y_1 + \cdots + x_ny_n = \sum_{k=1}^{n} x_k y_k$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

This is an inner product on $\mathbb{R}^n$, called the Euclidean inner product.
This inner product induces the norm

$$\|x\| = \sqrt{(x,x)} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

which is the usual length of vector x.

Ex: On $\mathbb{C}^n$, consider $(z,w) = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ where

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

This is an inner product. [Note that it wouldn't be an inner product without the complex conjugate signs.] The corresponding norm is

$$\|z\| = \sqrt{(z, z)}$$
$$= \sqrt{z_1 \bar{z}_1 + z_2 \bar{z}_2 + \dots + z_n \bar{z}_n}$$
$$= \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}.$$

Ex: On $M_{2\times 3}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$, how to define an inner product of two matrices?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

One can regard $M_{2\times 3}(F)$ as F^6. Define

$$(A, B) = a_{11}\bar{b}_{11} + a_{12}\bar{b}_{12} + a_{13}\bar{b}_{13} + a_{21}\bar{b}_{21} + a_{22}\bar{b}_{22} + a_{23}\bar{b}_{23}.$$

Ex: On $M_{m\times n}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$, define inner product as

$$(A, B) = \underbrace{a_{11}\bar{b}_{11} + a_{12}\bar{b}_{12} + \dots + a_{mn}\bar{b}_{mn}}_{\text{sum of } m\times n \text{ terms}}$$
$$= \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}\bar{b}_{jk}.$$
$$= \operatorname{trace}(B^* A)$$

where B^* is the matrix obtained by taking the transpose of B and then taking the conjugate. For example, if

$$B = \begin{bmatrix} i & 1 \\ 1-i & 0 \\ 2 & 3i \end{bmatrix} \quad \text{then} \quad B^T = \begin{bmatrix} i & 1-i & 2 \\ 1 & 0 & 3i \end{bmatrix}$$

and

$$B^* = \overline{B}^T = \begin{bmatrix} -i & 1+i & 2 \\ 1 & 0 & -3i \end{bmatrix}.$$

This inner product on $M_{m\times n}(F)$ is referred to as Frobenius inner product. Another name of its is Hilbert-Schmidt inner product. Regardless of the names, the inner product is quite simple to understand: one simply regard $M_{m\times n}(F)$ as F^{mn}. Then use the regular inner product on F^{mn}.

Ex: On $V = \{f : [0,1] \to F,\ f \text{ is continuous}\}$, we define

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

This is an inner product on V. Note that when $F = \mathbb{R}$ then one can ignore the complex conjugate sign. For $f(x) = x$ and $g(x) = x^2$, for example, we have

$$(f, g) = \int_0^1 x\, x^2\, dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$

Ex:

One can switch any abstract vector space into an inner product space. This is the spirit of Problem 4 of HW5. The idea is as follows. Let V be an abstract vector space. To equip V with an inner product, we first choose a basis of V, say $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$.

Then the inner product of u and v in V is defined as the inner product of $[u]_\mathcal{B}$ and $[v]_\mathcal{B}$ in F^n.

* More examples on norm:

On $\mathbb{R}^n$, there are many ways to define a norm. For each $p \geq 1$, let us define an operator $\|\cdot\|_p$ as follows

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p}.$$

This is known as a p-norm on $\mathbb{R}^n$.

The taxicab norm corresponds to $p=1$. The Euclidean norm corresponds to $p=2$. The proof of $\|\cdot\|_p$ being a norm is nontrivial. The difficulty lies in the triangle inequality. In fact, the proof of triangle inequality is tricky enough to receive a name: Minkowski inequality. It is the following inequality:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Except in the case $p=2$, the norm $\|\cdot\|_p$ doesn't come from any inner product. The reason is that the parallelogram identity is not satisfied:

$$\|x+y\|_p + \|x-y\|_p \neq 2(\|x\|_p + \|y\|_p).$$