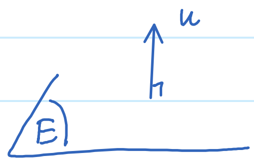
Lecture 21

Wednesday, February 26, 2020

An application of inner product is to define projections. Let V be an inner product space.

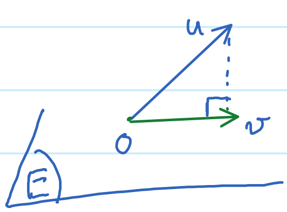
* <u>Definition</u>:

- Two vectors u and v are said to be perpendicular (or orthogonal) to each other (denoted by $u \perp v$) if $(u, v) = 0$.
- Let $u \in V$ and E be a subspace of V. Then u is said to be perpendicular (or orthogonal) to E (denoted by $u \perp E$) if u is perpendicular to every vector in E.



This definition is quite natural. We know from geometry that a vector is said to be perpendicular to a plane if it is perpendicular to every vector on the plane. The definition we just gave is a generalization of this concept. One now can talk about a vector being orthogonal to a subspace, not just a 2D-plane.

In 3D-geometry, one can talk about orthogonal projection of a vector on a plane. We can generalize this idea as follows.



* <u>Definition</u>:

Vector v is said to be the orthogonal projection of u on subspace E if two following conditions are satisfied:

(1) $v \in E$,

(2) $u - v \perp E$.

The question now is how to compute the projection of u on E. In other words, given u and E, we want to solve for vector $v \in E$ such that $u - v \perp E$. We acknowledge a difficulty: it is practically

To check if a vector is perpendicular to every single vector in a vector space E. A vector space has infinitely many vectors! However, we in fact only need to check if the vector is perpendicular to every vector in a basis. The following theorem explain this idea.

Theorem:

Let V be an inner product, E be a subspace of V, and $u \in V$, Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a basis of E. Then $u \perp E$ if and only if $u \perp v_k$ for all $k = 1, 2, \dots, n$.

Why is this true? The theorem says "if and only if". So there are two statements to check:

(a) Suppose $u \perp E$. Show that $u \perp v_k$ for all $k = 1, 2, \dots, n$.

(b) Suppose $u \perp v_k$ for all $k = 1, 2, \dots, n$. Show that $u \perp E$.

Part (a) is quite easy! Because $u \perp E$, u has to be perpendicular to every vector in E. Since $v_1, v_2, \dots, v_k$ are vectors in E, u has to be perpendicular to each of them.

Part (b) is more interesting to prove. We have

$$u \perp v_k \quad \forall k = 1, 2, \dots, n.$$

We want to show $u \perp E$. That is to show

$$u \perp v \quad \forall v \in E.$$

Let $v \in E$. We want to show $u \perp v$. That is to show $(u, v) = 0$. Because $v_1, v_2, \dots, v_n$ form a basis of E, v is a linear combination of these vectors. We can write

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_n \in F$. Then

$$\begin{aligned} (u, v) &= (u, c_1 v_1 + c_2 v_2 + \dots + c_n v_n) \\ &= (u, c_1 v_1) + (u, c_2 v_2) + \dots + (u, c_n v_n) \\ &= \bar{c}_1 \underbrace{(u, v_1)}_{0} + \bar{c}_2 \underbrace{(u, v_2)}_{0} + \dots + \bar{c}_n \underbrace{(u, v_n)}_{0} = 0. \end{aligned}$$

Thus $(u, v) = 0$. The theorem is proven. ■

We return to the question we asked earlier: given a vector $u \in V$ and a subspace $E \subset V$, how do we find the projection of u onto E? Let this projection be $v \in E$. We can write v as a linear combination of $v_1, v_2, \ldots, v_n$:

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

We want to solve for $c_1, c_2, \ldots, c_n \in F$ such that $u - v \perp E$.

The condition $u - v \perp E$ boils down to the condition

$$(u-v, v_1) = (u-v, v_2) = \cdots = (u-v, v_n) = 0.$$

The first term is equal to

$$\begin{aligned}(u-v, v_1) &= (u, v_1) - (v, v_1) \\ &= (u, v_1) - (c_1 v_1 + \cdots + c_n v_n, v_1) \\ &= (u, v_1) - c_1 (v_1, v_1) - c_2 (v_2, v_1) - \cdots - c_n (v_n, v_1).\end{aligned}$$

Since this is supposed to be zero, we get an equation

$$c_1 \underbrace{(v_1, v_1)}_{\text{known}} + c_2 \underbrace{(v_2, v_1)}_{\text{known}} + \cdots + c_n \underbrace{(v_n, v_1)}_{\text{known}} = \underbrace{(u, v_1)}_{\text{known}}.$$

Similarly, the equation $(u-v, v_2) = 0$ gives us an equation

$$c_1 \underbrace{(v_1, v_2)}_{\text{known}} + c_2 \underbrace{(v_2, v_2)}_{\text{known}} + \cdots + c_n \underbrace{(v_n, v_2)}_{\text{known}} = \underbrace{(u, v_2)}_{\text{known}}.$$

And so on. We eventually get a linear system of n equations and n unknowns.

If $\mathcal{B}$ is orthogonal, i.e any two vectors in $\mathcal{B}$ are perpendicular to each other, then the first equation becomes

$$c_1 (v_1, v_1) = (u, v_1).$$

The second equation becomes

$$c_2 (v_2, v_2) = (u, v_2).$$

And so on. In this case, the system becomes trivial to solve.

<u>Definition:</u> A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be orthogonal if any two vectors in this set are perpendicular to each other: $v_j \perp v_k$ for any $j \neq k$. If the set is orthogonal and $\|v_1\| = \|v_2\| = \ldots = \|v_n\| = 1$ then it is said to be orthonormal.

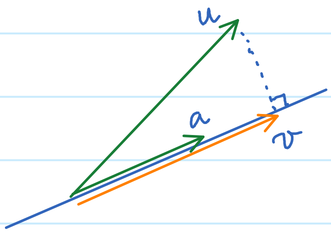
From our analysis above, if E has an orthogonal basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ then the project of $u \in V$ on E is given by

$$v = \text{proj}_E u = \frac{(u, v_1)}{\|v_1\|^2} v_1 + \frac{(u, v_2)}{\|v_2\|^2} v_2 + \cdots + \frac{(u, v_n)}{\|v_n\|^2} v_n .$$

Note that this formula is true <u>only if $\mathcal{B}$ is an orthogonal basis.</u>
Sometimes, for convenience we write $\text{proj}_{\{v_1, \ldots, v_n\}} u$ instead of $\text{proj}_E u$.

The notation $\text{proj}_{\{v_1, \ldots, v_n\}} u$ denotes the projection of vector u on the vector space spanned by $\{v_1, \ldots, v_n\}$.

<u>Ex</u>: Let $a \in V$ and $u \in V$. What is the projection of vector u on the line that contains a?



The line is the vector space spanned by the vector a. It has basis $\mathcal{B} = \{a\}$. This is an orthogonal basis since it contains only one element. Thus,

$$v = \text{proj}_{\{a\}} u = \frac{(u, a)}{\|a\|^2} a .$$