

Lecture 23

Monday, March 2, 2020

Orthogonal basis of a subspace E of an inner product space V makes the task of finding projection of a vector onto E simpler. For example, if $B = \{u_1, u_2, \dots, u_n\}$ is an orthogonal basis of E then the projection of any vector $u \in V$ onto E is given by

$$\text{proj}_E u = \frac{(u, u_1)}{(u_1, u_1)} u_1 + \frac{(u, u_2)}{(u_2, u_2)} u_2 + \dots + \frac{(u, u_n)}{(u_n, u_n)} u_n \quad (*)$$

If B is not an orthogonal basis, one either has to solve a system of equations to find the coefficients c_1, \dots, c_n of

$$\text{proj}_E u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

or orthogonalize B before using formula (*).

Ex: In $M_{2 \times 2}(\mathbb{R})$, consider the subspace

$$V = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

Find an orthogonal basis of V .

One can check that the vectors

$$u_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

are linearly independent. Thus, $B = \{u_1, u_2, u_3\}$ is a basis of V .

To find an orthogonal basis of V , we fix the vectors in B , one at a time.

There is no need to fix u_1 , so $v_1 = u_1$.

The first vector that needs to be fixed at all is u_2 .

$$v_2 = u_2 - \text{proj}_{\{u_1\}} u_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1.$$

Recall that $M_{2 \times 2}(\mathbb{R})$ is an inner product space with the natural inner product $(A, B) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$ (the Frobenius inner product).

We have $(u_2, v_1) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = 1(1) + 0(1) + 0(0) + (-1)(1) = 0.$

Thus, $v_2 = u_2 - \frac{0}{(v_1, v_1)} v_1 = u_2.$

In other words, there was no need to fix u_2 because u_2 is already perpendicular to $v_1 = u_1$. Now we fix u_3 :

$$v_3 = u_3 - \text{proj}_{\{v_1, v_2\}} u_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2 \quad (**)$$

We have

$$(u_3, v_1) = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = 1(1) + 0(1) + (-1)(0) + 1(1) = 2.$$

$$(u_3, v_2) = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1(1) + 0(0) + (-1)(0) + 1(-1) = 0.$$

$$(v_1, v_1) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = 1(1) + 1(1) + 0(0) + 1(1) = 3.$$

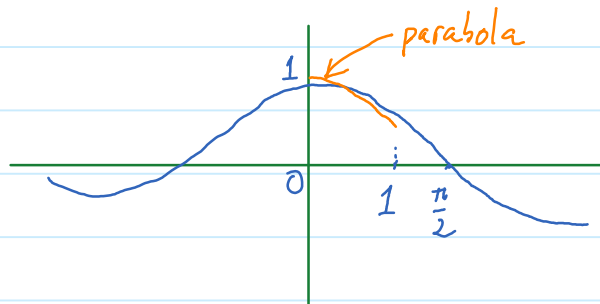
Thus, (**) becomes

$$v_3 = u_3 - \frac{2}{3} v_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ -1 & 1/3 \end{bmatrix}.$$

In conclusion, the basis $\mathcal{B} = \{v_1, v_2, v_3\}$ is an orthogonal basis of V .

* Optimization problem:

An important application of inner product spaces is to formulate and solve optimization problems. Let us consider an example.



The cosine function looks like a parabola on the interval $[0, \pi/2]$, but

It is not a parabola. (A parabola has second derivative equal to a constant, while the cosine curve has second derivative equal to another cosine curve!)

However, it is legitimate to ask what is the parabola that best approximates the cosine function on the interval $[0, 1]$? There are of course many ways to approximate the cosine function by a parabola. To decide which one is "the best", we have to find a way to quantify how close a parabola is to the cosine curve. This is where the notion of norm is helpful.

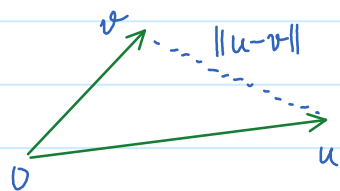
The vector space $V = \{u: [0, 1] \rightarrow \mathbb{R}, u \text{ is continuous}\}$ is an inner product space with

$$(u, v) = \int_0^1 u(x)v(x) dx.$$

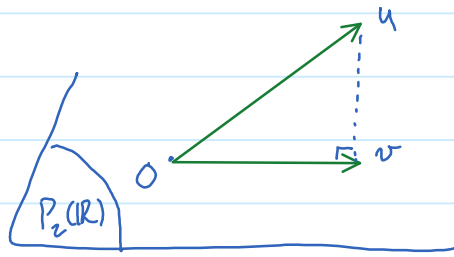
V is a very big vector space. It contains most functions that we are familiar with: the polynomials, exponentials, sine, cosine, ... The inner product on V induces a norm on V :

$$\|u\| = \sqrt{(u, u)} = \left(\int_0^1 u^2(x) dx \right)^{1/2}.$$

The distance between vector u and vector v is the length of vector $u-v$, which is $\|u-v\|$.



We want to find a parabola $v(x) = ax^2 + bx + c$ that is closest to $u(x) = \cos x$. That is to search for vector $v \in P_2(\mathbb{R})$ such that $\|u-v\|$ is minimized. Geometrically, v should be the projection of u on the space $P_2(\mathbb{R})$.



The problem becomes finding $\text{proj}_{P_2(\mathbb{R})} u$.

We have converted the optimization problem into a linear algebra problem. We will finish this problem next time.