Lecture 24

Wednesday, March 4, 2020

Recall that the vector space $V = \{u: [0,1] \to \mathbb{R},\ u \text{ is continuous}\}$ is an inner product space with inner product defined by

$$(u,v) = \int_0^1 u(x)\,v(x)\,dx.$$

This inner product induces a norm

$$\|u\| = \sqrt{(u,u)} = \left(\int_0^1 u^2(x)\,dx\right)^{1/2}.$$

This norm is sometimes referred to as an energy norm. When u is a velocity function then $\|u\|^2$ is proportional to the energy.

The problem of finding a parabola $v(x) = ax^2 + bx + c$ that best approximates the cosine function $u(x) = \cos x$ becomes the problem of finding the projection $v = \text{proj}_{P_2(\mathbb{R})} u$.

We know two ways to do this: solving a system of equations or orthogonalizing a basis. Let us choose the first way.

A basis of $P_2(\mathbb{R})$ is $\mathcal{B} = \{\underbrace{x^2}_{u_1}, \underbrace{x}_{u_2}, \underbrace{1}_{u_3}\}$. Then $v = c_1u_1 + c_2u_2 + c_3u_3$

where c_1, c_2, c_3 solve the matrix equation

$$\begin{bmatrix} (u_1,u_1) & (u_2,u_1) & (u_3,u_1) \\ (u_1,u_2) & (u_2,u_2) & (u_3,u_2) \\ (u_1,u_3) & (u_2,u_3) & (u_3,u_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (u,u_1) \\ (u,u_2) \\ (u,u_3) \end{bmatrix}$$

We have

$$(u_1,u_1) = (x^2,x^2) = \int_0^1 x^4\,dx = \frac{1}{5},$$

$$(u_2,u_1) = (x,x^2) = \int_0^1 x^3\,dx = \frac{1}{4},$$

$$(u_3, u_1) = (1, x^2) = \int_0^1 x^2\,dx = \frac{1}{3},$$

$$(u_2, u_2) = (x, x) = \int_0^1 x^2\,dx = \frac{1}{3},$$

$$(u_3, u_3) = (1, 1) = \int_0^1 1\,dx = 1.$$

$$(u, u_1) = \int_0^1 (\cos x)\, x^2\,dx = 2\cos 1 - \sin 1.$$

(Integration by part twice)

$$(u, u_2) = \int_0^1 (\cos x)\, x\,dx = \sin 1 + \cos 1 - 1$$

$$(u, u_3) = \int_0^1 \cos x\,dx = \sin 1.$$

The matrix equation now becomes

$$\underbrace{\begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2\cos 1 - \sin 1 \\ \sin 1 + \cos 1 - 1 \\ \sin 1 \end{bmatrix}}_{b}$$

Then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} b \approx \begin{bmatrix} -0.4310 \\ -0.0365 \\ 1.0034 \end{bmatrix}.$$

(One can use the command inv(A)*b in Matlab.)

Therefore, the parabola that best approximates the cosine function on the interval $[0,1]$ is

$$\begin{aligned} v(x) &= c_1 u_1 + c_2 u_2 + c_3 u_3 \\ &\approx -0.4310\, x^2 - 0.0365\, x + 1.0034. \end{aligned}$$

$\underline{\text{Ex}}$: Find an orthonormal basis of $P_2(\mathbb{R})$. The inner product is still given by $\quad (u,v) = \int_0^1 u(x)v(x)\,dx.$

A basis of $P_2(\mathbb{R})$ is $\mathcal{B} = \{\underbrace{x^2}_{u_1}, \underbrace{x}_{u_2}, \underbrace{1}_{u_3}\}$.

We use Gram-Schmidt procedure to orthogonalize $\mathcal{B}$:

$$v_1 = u_1 = x^2,$$

$$v_2 = u_2 - \operatorname{proj}_{\{v_1\}} u_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1.$$

Since $\quad (u_2, v_1) = \int_0^1 x^3\,dx = \frac{1}{4}$ and $(v_1, v_1) = \int_0^1 x^4\,dx = \frac{1}{5}$,

we get $\quad v_2 = x - \frac{1/4}{1/5}x^2 = x - \frac{5}{4}x^2.$

Then $\quad v_3 = u_3 - \operatorname{proj}_{\{v_1, v_2\}} u_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2.$

We have $\quad (u_3, v_1) = \int_0^1 x^2\,dx = \frac{1}{3}$

$$(u_3, u_2) = \int_0^1 x\,dx = \frac{1}{2}$$

$$(v_2, v_2) = \int_0^1 \left(x - \frac{5}{4}x^2\right)^2 dx = \frac{1}{48}$$

Thus, $\quad v_3 = 1 - \frac{1/3}{1/5}x^2 - \frac{1/2}{1/48}\left(x - \frac{5x^2}{4}\right)$

$$= 1 - 24x + \frac{85}{3}x^2.$$

We obtain an orthogonal basis of $P_2(\mathbb{R})$, namely
$$\mathcal{C} = \{v_1, v_2, v_3\}.$$

To get an orthonormal basis of $\mathcal{C}$, we rescale v_1, v_2, v_3 to make sure that they have length equal to 1.

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\langle v_1, v_1 \rangle}} v_1 = \frac{1}{\sqrt{1/5}} x^2 = \sqrt{5}\, x^2,$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\langle v_2, v_2 \rangle}} v_2 = \frac{1}{\sqrt{1/3}} \left(x - \frac{5x^2}{4}\right) = \sqrt{3}\left(x - \frac{5x^2}{4}\right),$$

$$w_3 = \frac{v_3}{\|v_3\|} = \dots$$

Then $\{w_1, w_2, w_3\}$ is an orthonormal basis of $P_2(\mathbb{R})$.