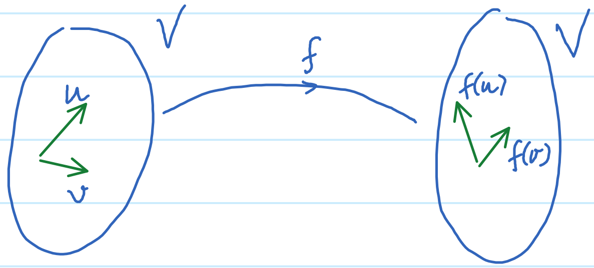
Lecture 26

Monday, March 9, 2020

We know that linear maps are compatible with the structure of a vector space. If vector space V is equipped with an inner product, making it an inner product space, one can ask: what kind of linear maps is "compatible" with the inner product on V?



The most natural answer is probably the linear maps that preserve inner products: $(f(u), f(v)) = (u,v)$ for all $u, v \in V$. Such a map is called a unitary map. We will analyze some geometric consequences of unitary maps as follows.

Let $f: V \to V$ be a unitary map. Then $(f(u), f(u)) = (u,u)$ for all $u \in V$. That is

$$\|f(u)\|^2 = \|u\|^2 \quad \forall u \in V.$$

Therefore, $\|f(u)\| = \|u\|$ for all $u \in V$. We see that a unitary map is length-preserving. In other words, the length of a vector doesn't change after applying f.

The "angle" between two abstract vectors u and v can be defined as θ satisfying

$$\cos\theta = \frac{(u,v)}{\|u\|\,\|v\|}.$$

The angle between vectors $f(u)$ and $f(v)$ is γ satisfying

$$\cos\gamma = \frac{(f(u), f(v))}{\|f(u)\|\,\|f(v)\|}.$$

Because $(f(u), f(v)) = (u,v)$ and $\|f(u)\| = \|u\|$, $\|f(v)\| = \|v\|$, we have $\cos\gamma = \cos\theta$. If we only allow angles to be between 0 and π then $\gamma = \theta$. Hence, a unitary map is angle preserving. In other words, the angle between two vectors doesn't change after applying f.

Ex: Let V be an inner product space and $f: V \to V$ be a linear map given by $f(v) = 2v$. Is f a unitary map?

Intuitively, one can perceive that f is not unitary because it doesn't preserve lengths (although it does preserve angles). In general, the scaling is not unitary unless the scaling factor is a complex number with modulus 1. One can of course write a more rigorous proof as follows.

$$(f(u), f(u)) = (2u, 2u) = 4(u,u)$$

which is not equal to (u,u) for any $u \neq 0$. Therefore, f is not unitary.

Unitary maps behave like rotations. In the content of $\mathbb{R}^2$ (or $\mathbb{R}^n$ in general), unitary maps are exactly the geometric rotations about the origin (which we are already familiar with).

The inner product on V gives any linear map $f: V \to V$ a companion operator, which we call adjoint operator, $f^*: V \to V$. Adjoint operators are a very helpful tool to study spectral theory (i.e. eigenvalue, eigenspace, ...) on inner product spaces.

Definition:

Let V be an inner product space and $f: V \to V$ be a linear map. Then a map $f^*: V \to V$ is called the adjoint operator of f if it satisfies the following property:

$$(f(u), v) = (u, f^*(v)) \quad \forall\, u, v \in V.$$

One can define adjoint operator of a linear map $f: V \to W$. But for now we are only interested in the case $V = W$ for simplicity. Let us consider a few examples of finding the adjoint of a linear map.

Ex: $V = \mathbb{R}^2$

$$f: \mathbb{R}^2 \to \mathbb{R}^2, \quad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-2y \end{bmatrix}.$$

Find f^*.

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$ be two general vectors.

Because f^* is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, we only need to know what $f^*(v)$ is. Let us write

$$f^*(v) = f^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$$

where c and d are to be determined in terms of a and b. By the definition of adjoint operators, we have

$$(f(u), v) = (u, f^*(v)).$$

$$LHS = \left(f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \begin{bmatrix} a \\ b \end{bmatrix}\right) = \left(\begin{bmatrix} x+y \\ x-2y \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right)$$

$$= (x+y)a + (x-2y)b.$$

$$RHS = \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = xc + yd.$$

We want to find c and d such that

$$(x+y)a + (x-2y)b = xc + yd.$$

One can rearrange terms on the left hand side as $x(a+b) + y(a-2b)$.

Therefore,

$$\begin{cases} c = a+b \\ d = a-2b \end{cases}$$

We conclude that f^* is the map given by $f^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-2b \end{bmatrix}$.

In this example, f^* is exactly the same as f.

Ex: $f: M_{2\times 2}(\mathbb{C}) \longrightarrow M_{2\times 2}(\mathbb{C})$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} ib & a \\ c & 0 \end{bmatrix}$$

Find f^*.

Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $v = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be general vectors in $M_{2\times 2}(\mathbb{C})$.

We want to find $f^*(v) = f^*\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\right)$.

Let us write $f^*\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\right) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are to be determined in terms of x, y, z, t. By the definition of adjoint operators,

$$(f(u), v) = (u, f^*(v)).$$

$$\text{LHS} = \left(f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right), \begin{bmatrix} x & y \\ z & t \end{bmatrix}\right) = \left(\begin{bmatrix} ib & a \\ c & 0 \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}\right)$$

$$= ib\bar{x} + a\bar{y} + c\bar{z} + 0\bar{t}$$

$$= ib\bar{x} + a\bar{y} + c\bar{z}.$$

$$\text{RHS} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma} + d\bar{\delta}.$$

In order to have

$$ib\bar{x} + a\bar{y} + c\bar{z} = a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma} + d\bar{\delta}$$

we need

$$\begin{cases} i\bar{x} = \bar{\beta}, \\ \bar{y} = \bar{\alpha} \\ \bar{z} = \bar{\gamma} \\ 0 = \bar{\delta} \end{cases} \quad \text{which implies} \quad \begin{cases} \beta = \bar{\bar{\beta}} = \overline{i\bar{x}} = \bar{i}\,\bar{\bar{x}} = -ix \\ \alpha = y \\ \gamma = z \\ \delta = 0. \end{cases}$$

Therefore, $$f^*\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix}\right) = \begin{bmatrix} y & -ix \\ z & 0 \end{bmatrix}.$$

Ex: $P_2(\mathbb{R})$ is equipped with inner product $\quad (u,v) = \int_0^1 u(x)\,v(x)\,dx.$

Consider the linear map
$$G: P_2(\mathbb{R}) \to P_2(\mathbb{R})$$
$$G(u)(x) = u(x+1)$$
Find G^*.

Let $u = ax^2+bx+c$ and $v = \alpha x^2+\beta x+\gamma$ be general vectors in $P_2(\mathbb{R})$.
We want to find $G^*(v)$. Let us write
$$G^*(v)(x) = mx^2+nx+p$$
where $m,n,p \in \mathbb{R}$ are to be determined in terms of α, β, γ.
By the definition of adjoint operators,
$$(G(u), v) = (u, G^*(v)).$$
$$\begin{aligned} LHS &= \left(a(x+1)^2 + b(x+1)+c,\ \alpha x^2+\beta x+\gamma\right) \\ &= \left(ax^2+(2a+b)x + a+b+c,\ \alpha x^2+\beta x+\gamma\right) \\ &= \int_0^1 \left(ax^2+(2a+b)x + a+b+c\right)(\alpha x^2+\beta x+\gamma)\,dx \\ &= \int_0^1 \Big\{a\alpha x^4 + [(2a+b)\alpha + a\beta]x^3 + [a\gamma + (2a+b)\beta + (a+b+c)\alpha]x^2 \\ &\qquad + [(a+b+c)\alpha + (2a+b)\gamma]x + (a+b+c)\gamma\Big\}\,dx \end{aligned}$$

$$\begin{aligned} LHS = \frac{1}{5}a\alpha + [(2a+b)\alpha + a\beta]\frac{1}{4} + [a\gamma+(2a+b)\beta + (a+b+c)\alpha]\frac{1}{3} \\ + [(a+b+c)\alpha + (2a+b)\gamma]\frac{1}{2} + (a+b+c)\gamma. \end{aligned}$$

$$\begin{aligned} RHS &= (ax^2+bx+c,\ mx^2+nx+p) \\ &= \int_0^1 (ax^2+bx+c)(mx^2+nx+p)\,dx \end{aligned}$$

$$= \int_0^1 [am x^4 + (an+bm)x^3 + (a\gamma + b\beta + c\alpha)x^2 + (b\gamma + c\beta)x + c\gamma]\, dx$$

$$RHS = \frac{1}{5}am + (an+bm)\frac{1}{4} + (ap+bn+cm)\frac{1}{3} + (bp+cn)\frac{1}{2} + cp.$$

We know that LHS = RHS for any a, b, c. Let us take $a=1, b=c=0$. The equation LHS = RHS becomes

$$\frac{23}{15}\alpha + \frac{11}{12}\beta + \frac{7}{3}\gamma = \frac{m}{5} + \frac{n}{4} + \frac{p}{3}. \qquad (1)$$

Let us take $a=0, b=1, c=0$. The equation LHS = RHS becomes

$$\frac{13}{12}\alpha + \frac{1}{3}\beta + \frac{3}{2}\gamma = \frac{m}{4} + \frac{n}{3} + \frac{p}{2}. \qquad (2)$$

Let us take $a=0, b=0, c=1$. The equation LHS = RHS becomes

$$\frac{5}{6}\alpha + \gamma = \frac{m}{3} + \frac{n}{2} + p. \qquad (3)$$

The system of equations (1), (2), (3) can be written in matrix form as

$$\begin{bmatrix} 23/15 & 11/12 & 7/3 \\ 13/12 & 1/3 & 3/2 \\ 5/6 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \\ p \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} m \\ n \\ p \end{bmatrix} = \begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 23/15 & 11/12 & 7/3 \\ 13/12 & 1/3 & 3/2 \\ 5/6 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 106 & 105 & 180 \\ -98 & -101 & -168 \\ 29/2 & 31/2 & 25 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

From here one can easily express m, n, p in terms of α, β, γ.