Lecture 3

Friday, January 10, 2020

Problem 1 on the worksheet:

$V = \{f : [0,1] \to \mathbb{R}, \text{ continuous}, f(1) = 0\}$.

Show that V is a vector space over $\mathbb{R}$

We need to check all 10 axioms (A0), (A1), ..., (D2). Let's check (A1).
We want to show that

$$f+g = g+f \qquad \forall f, g \in V.$$

Let $f, g \in V$. We want to show

$$f+g = g+f$$

That is to show

$$(f+g)(x) = (g+f)(x) \qquad \forall x \in [0,1]$$

By the definition of the sum of two functions, we can rewrite the left hand side as

$$(f+g)(x) = f(x) + g(x).$$

Similarly,

$$(g+f)(x) = g(x) + f(x).$$

We want to show

$$f(x) + g(x) = g(x) + f(x)$$

This is true because the addition of real numbers is commutative.

Problem 2 on the worksheet:

Show that $V = \mathbb{R}$ is not a vector space over $F = \mathbb{C}$

We only need to show that one of the 10 axioms is false. In particular, we will show that (S0) is false. (S0) states that

$$cv \in V \qquad \forall c \in F, v \in V.$$

Knowing that $F = \mathbb{R}$ and $V = \mathbb{C}$, let us change the notation:
(S0) says

$$zx \in \mathbb{R} \qquad \forall z \in \mathbb{C}, x \in \mathbb{R}$$

To show that this is not true, we give a counterexample. Let $z = i$ and $c = 1$. Then $zc = i \notin \mathbb{R}$

$\ast$ Definition:

Let W be a vector space over F. A subset $V \subset W$ is said to be a subspace of W if V, with the addition and scaling rules inherited from W, is a vector space over F.

Simply put, a subspace of W is a subset that is also a vector space.

So far, the only method we know to check if a set V is a vector space is to check 10 axioms. If we know that V is contained in some vector space then there are only 3 things we need to check.

$\ast$ Theorem:

Let W be a vector space over F and $V \subset W$ is a subset of W. Then V is a vector space over F (in other words, V is a subspace of W) if and only if the following conditions are satisfied:

(1) $0 \in V$,

(2) (A0): V is closed under addition,

(3) (S0): V is closed under scaling.

Example: the set $V = \{f : [0,1] \to \mathbb{R},\ f \text{ is continuous},\ f(1) = 0\}$ is a subset of $W = \mathbb{R}^{[0,1]}$.

We know that W is a vector space over $\mathbb{R}$. To check if V is a vector space over $\mathbb{R}$, we only need to check 3 things:

(1) Check if $0 \in V$.

The constant function 0 belongs to V because it is continuous and vanishes at 1.

(2) Check if V is closed under addition:

Let $f, g \in V$. We want to check if $h = f + g$ also belongs to V. We know from Calculus I that the sum of two continuous function is continuous. Thus, h is continuous. Moreover,

$$h(1) = (f+g)(1) = f(1) + g(1) = 0 + 0 = 0$$

Therefore, $h \in V$.

(3) Check if V is closed under scaling:

[Similarly]

Like in $\mathbb{R}^n$, we can define the notion of linear combination, linear independence, basis, dimension, ... on a general vector space.

Let V be a vector space over F. Consider vectors $v_1, v_2, \dots, v_n \in V$ and numbers $c_1, c_2, \dots, c_n \in F$. Then the vector

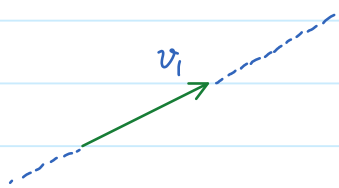
$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

is called a linear combination of $v_1, v_2, \dots, v_n$. The set of all such linear combinations is called a spanning set.

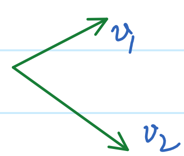
$$\text{span}\{v_1, v_2, \dots, v_n\} = \{c_1 v_1 + \dots + c_n v_n : c_1, c_2, \dots, c_n \in F\}.$$

In $\mathbb{R}^n$,

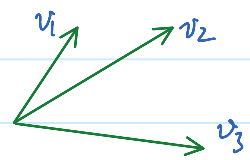
$\text{span}\{v_1\}$ is a line.



$\text{span}\{v_1, v_2\}$ is a plane



$\mathrm{span}\{v_1, v_2, v_3\}$ is a 3 dimensional space.



The span of a small set is a very big set

The function $f(x) = \sin x$ is a vector in the vector space $\mathbb{R}^{\mathbb{R}}$ Although $\mathbb{R}^{\mathbb{R}}$ is quite an abstract vector space, we can still visualize the set

$$\mathrm{span}\{f\} = \{cf : c \in \mathbb{R}\}$$

as a line parallel to f.