# Lecture 4

Monday, January 13, 2020

Last time, we defined linear combinations of vectors and spanning sets

We can also define linear independence of vectors.

* Definition:

Let $v_1, v_2, \dots, v_n$ be vectors in $V$, which is a vector space over $F$. These vectors are said to be linearly independent if the equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

with unknowns $c_1, c_2, \dots, c_n \in F$ has only one solution $c_1 = c_2 = \dots = c_n = 0$.

We can also define linear independence of a set of <u>infintely many</u> vectors

Let $S \subset V$ be an infinite subset. Then $S$ is said to be linearly independent if every finite subset of $S$ is linearly independent.

With the notion of linear independence and spanning set, we can define basis and dimension of a vector space

Let $V$ be a vector space over $F$. A subset $B \subset V$ is called a basis of $V$ if

$$\begin{cases} B \text{ is linearly independent,} \\ B \text{ spans } V\text{, i.e. } V = \operatorname{span} B. \end{cases}$$

The second property says that $B$ is big enough to span $V$, i.e. every vector in $V$ can be "generated" from vectors in $B$ (written as a linear combination of vectors in $B$).

The first property says that $B$ is small enough that there is no redundancy in $B$: no vector in $B$ can be "generated" by other vectors in $B$

The dimension of $V$ is defined as the number of vectors in a basis of $V$ and is denoted by $\dim_F V$ or simply $\dim V$ where $F$ is understood

Let us consider two well-known examples.

(1) $V = M_{m\times n}(F)$, the space of all matrices of size $m\times n$ with coefficients in $F$

Any vector in $V$ is a matrix of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

which can be written as

$$A = a_{11}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \cdots + a_{mn}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Put

$$E_{ij} = \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \begin{array}{l} \to i \\ \downarrow j \end{array}$$

(matrix with coefficient 1 at row $i$, column $j$, and 0 elsewhere)

Then

$$A = a_{11}E_{11} + a_{12}E_{12} + \cdots + a_{mn}E_{mn}.$$

Thus, $A$ is a linear combination of the set

$$S = \{E_{11}, E_{12}, \ldots, E_{mn}\}$$

Because $S$ is a subset of $V$ and every vector of $V$ is a linear combination of $S$, we have $V = \operatorname{span} S$

Is $S$ a basis of $V$? We need to check whether $S$ is linearly independent.

Consider the equation

$$a_{11}E_{11} + a_{12}E_{12} + \cdots + a_{mn}E_{mn} = \underbrace{0}_{\substack{\text{matrix of size } m\times n, \\ \text{with all coefficients equal to } 0}}$$

with unknowns $a_{11}, a_{12}, \ldots, a_{mn} \in F$.

We see that this equation is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

This gives $a_{11} = a_{12} = \cdots = a_{mn} = 0$

We have showed that $S$ is linearly independent. Therefore, $S$ is a basis of $V$. The set $S = \{E_{11}, E_{12}, \dots, E_{mn}\}$ is called the standard basis of $M_{m\times n}(F)$.

$$\dim_F V = \# S = mn.$$

② Denote by $P_n(F)$ the set of all polynomials of coefficients in $F$ with degree $\leq n$

Since a polynomial in $P_n(F)$ is also a function from $F$ to $F$, we have

$$P_n(F) \subset F^F.$$

The latter is a vector space. One can verify three criteria of subspace to show that $P_n(F)$ is a vector space over $F$.

How about basis and dimension of $P_n(F)$?

An element of $P_n(F)$ is of the form

$$u(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some $a_n, a_{n-1}, \dots, a_0 \in F$.

We see that $u$ is a linear combination of the nomials $x^n, x^{n-1}, \dots, x, 1$

In other words, any vector in $V = P_n(F)$ is a linear combination of the set

$$S = \{x^n, x^{n-1}, \dots, x, 1\}.$$

This suggests that $V = \operatorname{span} S$.

Is $S$ a basis of $V$? We need to check if $S$ is linearly independent

Consider the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad (1)$$

with unknowns $a_n, a_{n-1}, \dots, a_1, a_0 \in F$

Note that (1) is an equation of functions: two functions are equal if they are equal at every point, One can rewrite (1) as

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0 \quad \forall x \in F.$$

Recall that $F$ can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In any case, $F$ is an infinite set. The polynomial on the LHS, therefore, has infinitely many roots (every number in $F$ is a root of it). This is only possible if every coefficient of the polynomial is equal to 0. Hence,

$$a_n = a_{n-1} = \dots = a_1 = a_0 = 0$$

We have showed that $S$ is linearly independent. It is therefore a basis of $V = P_n(F)$.

$$S = \{x^n, x^{n-1}, \dots, x, 1\}$$

is called the standard basis of $P_n(F)$.

$$\dim_F V = \# S = n+1.$$

* Example: (first problem on worksheet)

$$V = \{u \in P_3(\mathbb{R}) : u(0) = u(1) = 0\}$$

Find a basis and the dimension of $V$.

The problem doesn't ask us to show that $V$ is a vector space. If we want to show that $V$ is a vector space, notice that $V$ is a subset of $P_3(\mathbb{R})$, which is known to be a vector space. Also, $V$ is closed under addition and scaling (needs to check carefully). Thus, $V$ is a subspace of $P_3(\mathbb{R})$.

To find a basis of $V$, let's consider a general element of $V$.

$$u(x) = ax^3 + bx^2 + cx + d$$

where $a, b, c, d \in \mathbb{R}$. The constraints $u(0) = 0$ and $u(1) = 0$ amounts to two equations:

$$d = 0, \qquad a + b + c + d = 0$$

We simplify as

$$d = 0, \quad c = -a-b$$

Here $a$ and $b$ are free variables in $\mathbb{R}$. Then we can rewrite $u$ as

$$u(x) = ax^3 + bx^2 + (-a-b)x$$
$$= a(x^3 - x^2) + b(x^2 - x)$$

which is a linear combination of $u_1(x) = x^3 - x^2$ and $u_2(x) = x^2 - x$. In other words, any vector of $V$ is a linear combination of the set $B = \{u_1, u_2\}$ we have showed that

$$V = \operatorname{span} B.$$

We will check if $B$ is linearly independent.

Consider the equation $au_1 + bu_2 = 0$ with unknowns $a, b \in \mathbb{R}$. This is an equation of functions. We can rewrite it as

$$a\,u_1(x) + b\,u_2(x) = 0 \quad \forall x \in \mathbb{R}.$$

Equivalently,

$$a(x^3 - x) + b(x^2 - x) = 0 \quad \forall x \in \mathbb{R}. \qquad (1)$$

There are two ways to show that $a = b = 0$.

* Method 1:

The equation can be written as

$$ax^3 + bx^2 + (-a-b)x = 0 \quad \forall x \in \mathbb{R}.$$

The LHS is a polynomial of degree $\leq 3$. It has infinitely many roots only if all coefficients are equal to zero. Thus,

$$a = b = -a-b = 0.$$

* Method 2:

Because equation (1) is true for all $x \in \mathbb{R}$, we can plug a few values of $x$:

For $x = 2$, we get $6a + 2b = 0$.

For $x = -1$, we get $2b = 0$.

From these two equations, we get $a = b = 0$.

In conclusion, $B = \{u_1, u_2\}$ is a basis of $V$ and $\dim V = 2$.